

Introduction to Khovanov Homologies

I. Unreduced Jones superpolynomial

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ABSTRACT

An elementary introduction to Khovanov construction of superpolynomials. Despite its technical complexity, this method remains the only source of a *definition* of superpolynomials from the first principles and therefore is important for development and testing of alternative approaches. In this first part of the review series we concentrate on the most transparent and unambiguous part of the story: the unreduced Jones superpolynomials in the fundamental representation and consider the 2-strand braids as the main example. Already for the 5_1 knot the unreduced superpolynomial contains more items than the ordinary Jones.

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Introduction

Knot polynomials – the Wilson-loop averages in (refined) $3d$ Chern-Simons theory [1, 2] – are once again coming to the avant-scene of theoretical physics, this time because they are at the intersection of a number of modern technical developments and look as potential members of various non-trivial dualities.

Relatively well understood are the HOMFLY polynomials, basically because there is powerful formulation in terms of the universal quantum \mathcal{R} -matrices, which allows to pose and answer a large variety of questions, at least in principle. At the same time, there is a more general class of *superpolynomials*, for which a similarly adequate formulation is still lacking. Instead there is a whole variety of different approaches [3]-[20], which give consistent results, but are still applicable only for particular classes of knots and links. The only exception at the moment is the original Khovanov homology method [21]-[37], which actually provides a general definition (with relatively modest ambiguity). Ironically, it is technically the most tricky and tedious, and its relation to other approaches is often obscure and difficult to establish. In addition, due to rather sophisticated presentation it is unfamiliar to many researchers with physical background. There was already a number of attempts to present Khovanov's approach in simple and transparent way: especially nice, in our opinion, is the wonderful Bar-Natan's paper [22]. This review note is just one more try, in fact, not very different from [22].

In this note – presumably the first in a series – we discuss the simplest example of Khovanov's calculus: for *unreduced* Jones superpolynomials. "Jones" means that we are restricted to $Sl(2)$ gauge group, parameter A of HOMFLY polynomials is restricted to $A = q^2$. Therefore we do not touch here the issue like matrix factorization, relevant for Khovanov-Rozansky deformation for arbitrary $A = q^N$. "Unreduced" means that T -deformed is the Jones polynomial itself, not divided by the quantum dimension. Unreduced superpolynomials are in many respects more complicated and less interesting than the ordinary (reduced) ones, but in Khovanov approach they are simpler, moreover, the difference from HOMFLY, arising after T -deformation, is clearly seen for them already at the level of the simplest 2-strand braids.

We assume some knowledge of the theory of HOMFLY polynomials, at least of its spirit, say, at the level of [38]-[43], and do not go into detail of the corresponding motivations. Instead we proceed directly to Khovanov's method and describe it in seven steps. The first four of them involve reformulation of Jones polynomial as Euler characteristic of a complex, associated with the knot/link diagram Γ , which at the fifth step is generalized to a Poincare polynomial. The central for these steps are the following *formulas*:

Step 1: Kauffman's \mathcal{R} -matrix [44]

$$\begin{aligned}\mathcal{R}_{kl}^{ij} &= q \left(\delta_k^i \delta_l^j - q \epsilon^{ij} \epsilon_{kl} \right), \\ (\mathcal{R}^{-1})_{kl}^{ij} &= -q^{-2} \left(\epsilon^{ij} \epsilon_{kl} - q \delta_k^i \delta_l^j \right), \\ \delta_i^i &= D = q + q^{-1}\end{aligned}\tag{1}$$

Step 2: Written in terms of this \mathcal{R} -matrix, Jones polynomial acquires a form

$$J(q) \sim \sum_{\text{resolutions of } \Gamma} (-q)^{|r-r_c|} D^{\nu(r)}\tag{2}$$

Step 3: On the set of resolutions one can define an action of cut-and-join operator

$$W^\Gamma = \frac{1}{2} \sum_{a,b,c} N_{bc}^a \left(p_a \frac{\partial^2}{\partial p_b \partial p_c} + p_b p_c \frac{\partial}{\partial p_a} \right)\tag{3}$$

where the time-variables p_a are associated with the cycles, appearing in the resolutions of Γ . This operator acts on *extended* Jones polynomials and converts them into polynomials for graphs with adjacent colorings.

Step 4: If time-variables are Miwa transformed, auxiliary q -graded vector spaces $V_a \cong V$ appear. After that the Jones polynomial can be rewritten as an Euler characteristic of a hypercube quiver

$$J(q) \sim \sum_I (-q)^I \dim_q C_I, \quad C_I = \bigoplus_{\substack{\text{resolutions of } \Gamma \\ \text{with a given } |r-r_c|=I}} V^{\otimes \nu(r)}\tag{4}$$

Step 5: Superpolynomial is defined as Poincare polynomial of associated complex

$$P(T, q) = \sum_I (qT)^I \left(\dim_q \text{Ker}(d_{I+1}) - \dim_q \text{Im}(d_I) \right)\tag{5}$$

Step 6: The differentials are provided by the BRST operator, which can be considered as a supersymmetrization of *a half* of the cut-and-join operator (3):

$$\mathfrak{Q}^\Gamma = \sum_{\substack{a \\ b < c}} \epsilon_{bc}^a N_{bc}^a \left(Q_{ij}^k \vartheta_a^i \frac{\partial^2}{\partial \vartheta_b^j \partial \vartheta_c^k} + Q_k^{ij} \vartheta_b^j \vartheta_c^k \frac{\partial}{\partial \vartheta_a^i} \right) \quad (6)$$

constructed with the help of the structure constants Q of commutative associative algebra. Particular differentials are obtained by picking up some items from \mathfrak{Q}^Γ , which are selected by the shape of extended Jones polynomial, introduced at the step 3 above.

Step 7: Evaluation of the cohomologies of d_I . At the present level of our understanding this last step can be made only in particular examples, for which we choose the simplest possible one: the 2-strand braids, giving the simplest possible torus links and knots, including the Hopf link and the trefoil. For all of them unreduced Jones polynomial are just 4-term polynomials, but already for the knot 5_1 the unreduced Jones superpolynomial contains more items (this is not the case for reduced superpolynomials: there the number of terms becomes different and new information is provided by T -deformation starting from higher number of intersections).

1 step: Locality principle and Kauffman's \mathcal{R} -matrix

Usually the classical \mathcal{R} -matrix in the fundamental representation of $Sl(2)$,

$$\mathcal{R}_{kl}^{ij} = \left(\begin{array}{c|cc|c} 1 & & & \\ \hline & 0 & 1 & \\ \hline & 1 & 0 & \\ \hline & & & 1 \end{array} \right) = \delta_l^i \delta_k^j = -\epsilon^{ij} \epsilon_{kl} + \delta_k^i \delta_l^j \quad (7)$$

is q -deformed into the $Sl_q(2)$ \mathcal{R} -matrix

$$\mathcal{R} = \left(\begin{array}{c|cc|c} q & & & \\ \hline & q - q^{-1} & 1 & \\ \hline & 1 & 0 & \\ \hline & & & q \end{array} \right) \quad (8)$$

which has triple eigenvalue q and one $-q^{-1}$, associated with the symmetric and antisymmetric channels [2] and [11] in decomposition of the tensor product $[1] \otimes [1] = [2] + [11]$.

There is, however, another deformation [44]:

$$\boxed{\mathcal{R}_{kl}^{ij} = q \left(\delta_k^i \delta_l^j - q \epsilon^{ij} \epsilon_{kl} \right)} \quad (9)$$

$$\left(\mathcal{R}^{-1} \right)_{kl}^{ij} = -q^{-2} \left(\epsilon^{ij} \epsilon_{kl} - q \delta_k^i \delta_l^j \right)$$

where indices $i, j, k, l = 1, \dots, D$ and dimension D is analytically continued to $D = [2]_q = q + q^{-1}$. In practice this means that $\epsilon^{ij} \epsilon_{kj} = \delta_k^i$ and $\epsilon^{ij} \epsilon_{ij} = \delta_i^i = D$. (In fact this rule implies that one can change $\epsilon^{ij} \epsilon_{kl}$ for $\delta^{ij} \delta_{kl}$ in all the expressions for link polynomials – this is not the same in components, but gives the same expression for traces of \mathcal{R} -matrix products.) The real role of ϵ 's is to account for orientation change:

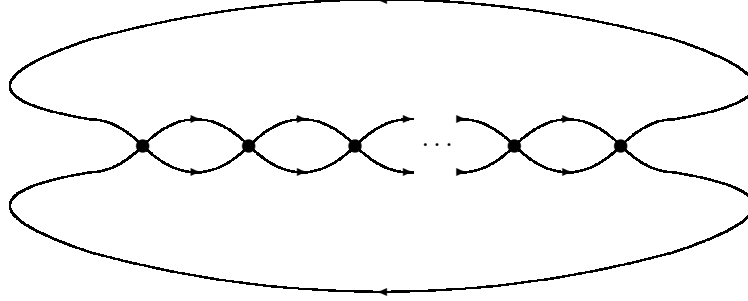
$$\begin{array}{c} \nearrow \searrow \\ \nwarrow \nearrow \end{array} = \begin{array}{c} \nearrow \searrow \\ \nwarrow \nearrow \end{array} \text{ with a dot at the intersection} = q \left(\begin{array}{c} \frown \smile \\ \smile \frown \end{array} \text{ with dots} - q^2 \begin{array}{c} \frown \smile \\ \smile \frown \end{array} \right)$$

For this choice of D the \mathcal{R} -matrix satisfies three Reidemeister conditions and skein (Hecke algebra) relation:¹

$$\begin{aligned}
R_1 : \quad & \mathcal{R}_{jk}^{ik} = \delta_j^i = \left(\mathcal{R}^{-1} \right)_{jk}^{ik}, \\
R_2 : \quad & \mathcal{R}_{kl}^{ij} \left(\mathcal{R}^{-1} \right)_{mn}^{kl} = \delta_m^i \delta_n^j, \\
R_3 : \quad & \left(\mathcal{R}^{-1} \right)_{bc}^{jk} \left(\mathcal{R}^{\pm 1} \right)_{le}^{ib} \mathcal{R}_{mn}^{ec} = \mathcal{R}_{ab}^{ij} \left(\mathcal{R}^{\pm 1} \right)_{en}^{bk} \left(\mathcal{R}^{-1} \right)_{lm}^{ae}, \\
\text{skein} : \quad & q^{-2} \mathcal{R}_{kl}^{ij} - q^2 \left(\mathcal{R}^{-1} \right)_{kl}^{ij} = -(q - q^{-1}) \delta_k^i \delta_l^j
\end{aligned} \tag{10}$$

This means that such \mathcal{R} can be used to evaluate the unreduced Jones polynomial for arbitrary like diagram, i.e. the Wilson-loop average in 3d Chern-Simons theory with the gauge group $Sl(2)$ [1] in the temporal gauge $A_0 = 0$ [42]. Projection of a knot/link on a 2d plane provides a link diagram, which is a planar 4-valent graph with two types of vertices, which we denote as black and white. One puts \mathcal{R} at black vertices and \mathcal{R}^{-1} at the white ones and contracts indices along all edges. This provides a knot polynomial (Jones in the fundamental representation for above choice of \mathcal{R}), which is independent of the projection – a topological invariant. Invariance is a corollary of Reidemeister relations, and the possibility to apply them is that above construction – contraction of \mathcal{R} matrices – is *local*: one can apply an identity, valid for several concrete vertices, and it will continue to hold for entire graph. This locality property looks absolutely trivial in this formulation, but it turns into not-quite-a-trivial-theorem in reformulations below. What is important, *locality* is the property of the *construction*, it does not depend on the nature of particular invariance: one could use locality with whatever one likes, not obligatory Reidemeister moves. On the contrary, Reidemeister invariance is a property of particular \mathcal{R} -matrix, and it is a *local* condition, imposed at the level of one, two, and three vertices – after that *locality* allows to extend it to arbitrary graphs.

For example, for 2-strand braid with n crossings (it is a knot and a two-component link for n odd and even respectively) we have (the picture is rotated by 90° to save the space, the same is done in the first part of the formula):



$$J_{\square}^{(n)}(q) = \text{Tr } \mathcal{R}^n = \underbrace{\mathcal{R}_{kl}^{ij} \mathcal{R}_{mn}^{kl} \dots \mathcal{R}_{ij}^{pq}}_n = q^n \text{Tr} \left\{ \left(\begin{array}{c|c} | & | \\ \hline - & q \end{array} \right)^n \right\} = q^n \left\{ \begin{array}{c} O \\ O \end{array} - nq \cdot O + \frac{n(n-1)}{2} q^2 \cdot O^2 + \dots \right\} =$$

¹Of all these it can deserve writing the $R3$ relation in more detail. This is the Yang-Baxter equation for braids, which is normally written as

$$\mathcal{R}_{32} \mathcal{R}_{12}^{\pm 1} \mathcal{R}_{23} = \mathcal{R}_{12} \mathcal{R}_{23}^{\pm 1} \mathcal{R}_{32}$$

or, in components,

$$\delta_a^i \left(\mathcal{R}^{-1} \right)_{bc}^{jk} \left(\mathcal{R}^{\pm 1} \right)_{de}^{ab} \delta_f^c \delta_l^d \mathcal{R}_{mn}^{ef} = \mathcal{R}_{ab}^{ij} \delta_c^k \delta_d^a \left(\mathcal{R}^{\pm 1} \right)_{ef}^{bc} \left(\mathcal{R}^{-1} \right)_{lm}^{de} \delta_n^f$$

Converting some indices we get:

$$\left(\mathcal{R}^{-1} \right)_{bc}^{jk} \left(\mathcal{R}^{\pm 1} \right)_{le}^{ib} \mathcal{R}_{mn}^{ec} = \mathcal{R}_{ab}^{ij} \left(\mathcal{R}^{\pm 1} \right)_{en}^{bk} \left(\mathcal{R}^{-1} \right)_{lm}^{ae}$$

Substitution of explicit expressions (9) for \mathcal{R} -matrices gives:

$$\begin{aligned}
\left(\delta_b^j \delta_c^k - q \delta^{jk} \delta_{bc} \right) \delta_l^i \delta_e^b \left(\delta^{ec} \delta_{mn} - q \delta_m^e \delta_n^c \right) &= \left(\delta^{ij} \delta_{ab} - q \delta_a^i \delta_b^j \right) \delta_e^{bk} \delta_{en} \left(\delta_l^a \delta_m^e - q \delta^{ae} \delta_{lm} \right), \\
\left(\delta_b^j \delta_c^k - q \delta^{jk} \delta_{bc} \right) \delta_l^i \delta_e^b \left(\delta^{ec} \delta_{mn} - q \delta_m^e \delta_n^c \right) &= \left(\delta^{ij} \delta_{ab} - q \delta_a^i \delta_b^j \right) \delta_e^{bk} \delta_n^b \left(\delta_l^a \delta_m^e - q \delta^{ae} \delta_{lm} \right)
\end{aligned}$$

and both relations should hold independently, because intermediate \mathcal{R} -matrix could enter in both powers +1 and -1. The first of these relations is trivially satisfied. The second one states

$$\delta_l^i \left((1 - qD + q^2) \delta^{jk} \delta_{mn} - q \delta_m^j \delta_n^k \right) = \delta_n^k \left((1 - qD + q^2) \delta^{ij} \delta_{lm} - q \delta_l^i \delta_m^j \right)$$

what is true, provided $D = q + q^{-1} = [2]_q$.

$$= q^n (D^2 - 1 + (1 - qD)^n) = q^n (q^{-2} + 1 + q^2 + (-q^2)^n) \quad (11)$$

where the circle diagram stands for the trace of unity, $\text{Tr } I = D = q + q^{-1}$. Similarly,

$$\text{Tr} (\mathcal{R}^{-1})^n = \left(-\frac{1}{q^2} \right)^n \left\{ (D - q)^n + (-q)^n (D^2 - 1) \right\} = \frac{1}{q^n} \left\{ q^2 + 1 + q^{-2} + \left(-\frac{1}{q^2} \right)^n \right\} = J_{\square}^{(n)}(q^{-1}) \quad (12)$$

The two expressions are related by Z_2 -symmetry $q \rightarrow q^{-1}$. One can notice that for any n the polynomials at the r.h.s. are divisible by $D = J_{\square}^{\text{unknot}}$ (though in a different ways for knots and links), what allows to introduce the *reduced* Jones polynomial $\check{J}_R = J_R / J_R^{\text{unknot}}$, which plays a big role in the theory of knot polynomials, but will be not discussed in the present paper. As for all torus knots, the *unreduced* Jones polynomial consists of just four items (this property gets obvious in alternative – matrix model – representation, see the last paper in [39] and the very first in [43]).

2 step: The double polynomial and the cube of resolutions

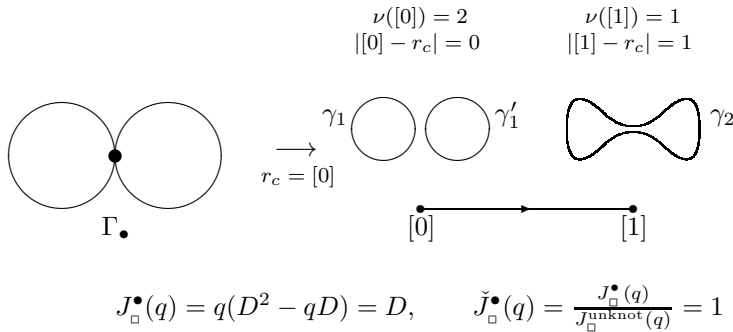
Concrete form of the Kauffman \mathcal{R} -matrix (9) implies that the Jones polynomial is actually a polynomial in two (related) variables q and D , which possesses a nice pictorial representation, which we actually used in the above example. The convoluted product of \mathcal{R} -matrices for a planar oriented 4-valent graph Γ_c with black and white vertices² is represented as a sum over all possible *resolutions* r of the graph:

$$J_{\square}^{\Gamma_c}(q) = (-)^{n_{\circ}} q^{n_{\bullet} - 2n_{\circ}} \sum_r^{2^n} (-q)^{|r - r_c|} D^{\nu(r)} \quad (13)$$

Here $n = n_{\bullet} + n_{\circ}$ is the number of vertices in Γ_c , and each of the 2^n resolutions contributes a product of two factors: $D^{\nu(r)}$, where $\nu(r)$ is the number of connected components in the resolved graph, and $(-q)^{|r - r_c|}$ where $|r - r_c|$ is the number of flips, needed to achieve the given resolution r from the original r_c , obtained when all black vertices are represented by $\big| \big|$ and all white vertices – by --- . This distance is the only place in this approach, where the colors of vertices play a role: changing colors imply that we count the number of flips $|r - r_c|$ from another r_c .

Thus what matters in (13) is the net of resolutions, connected by elementary flips $\big| \big| \leftrightarrow \text{---}$. Clearly, this net is nothing but an n -dimensional hypercube with 2^n vertices and $2^{n-1}n$ edges. Vertices are labeled by n -digit binary numbers $\alpha = [\alpha_1 \alpha_2 \dots \alpha_n]$, with $\alpha_1, \dots, \alpha_n = 0, 1$. If we forget the colors of vertices, $\Gamma_c \rightarrow \Gamma$ and enumerate the vertices of the graph Γ , then over the vertex α we put a particular resolution of the graph Γ , where at I -th vertex ($I = 1, \dots, n$) we put $\big| \big|$ if $\alpha_I = 0$ and put --- if $\alpha_I = 1$. The n edges, connecting a given vertex of the hypercube to adjacent ones, describe elementary flips at arbitrary vertex J of Γ . Original graph Γ_c with colored vertices defines a distinguished vertex in the hypercube with a resolution r_c , then $|r - r_c| = \sum_i |\alpha_I - \alpha_I^c|$, and now all the edges acquire orientation: they point away from r_c , and this is a step towards defining a quiver structure on the hypercube (a very simple one, in fact – just brushing from one vertex of the opposite).

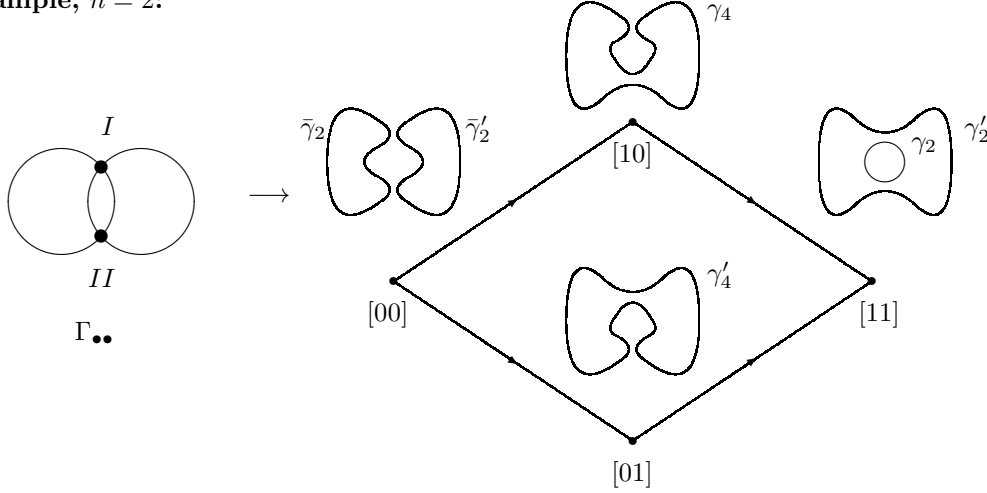
Example, $n = 1$:



²In knot-theory applications the graphs have all these properties, but many elements of the construction are more general: the graphs need not be planar, and also can have more types of vertices of different valencies, though in this case *resolutions* will be substituted by other notions.

$$\begin{array}{c}
\nu([0]) = 2 \quad \nu([1]) = 1 \\
|[0] - r_c| = 1 \quad |[1] - r_c| = 0 \\
\begin{array}{ccc}
\Gamma_o & \xrightarrow{r_c = [1]} & \begin{array}{c} \gamma_1 \quad \gamma'_1 \quad \gamma_2 \\ \text{---} [0] \text{---} [1] \end{array}
\end{array} \\
J_\square^\circ(q) = -q^{-2}(D - qD^2) = D, \quad \check{J}_\square^\circ(q) = \frac{J_\square^\circ(q)}{J_{\text{unknot}}^\circ(q)} = 1
\end{array}$$

Example, $n = 2$:



In Appendix A at the end of this paper we explain, how the topological invariance of the answer (13) for Jones polynomial can be proved in terms of this hypercube construction – without reference to the algebraic proof in sec.1. Since algebraic proof exists and is far more straightforward, there is no direct need to consider more sophisticated alternatives – but they help to highlight other sides of the theory, thus we include this alternative proof, but aside of the main line of our presentation. As a clarifying example, in Appendix A we discuss briefly what happens if one attempts to include the third (“ u -channel”) resolution: the result remains invariant under the two of the three Reidemeister shifts $E1$ and $R2$, but $R3$ -invariance is violated (at the \mathcal{R} -matrix language of sec.1 this simply means that the Yang-Baxter relation is violated).

Given (13), it is a natural idea to release q -dependence of D and treat q and D as independent parameters. Unfortunately, it does not work so simple: if we do this directly in (13) topological invariance under Reidemeister moves will be lost: say,

$$\pi(t, D) = \sum_r^{2^n} t^{|r - r_c|} D^{\nu(r)} \tag{14}$$

is not an invariant. However, after profound modification of (13) this idea turns viable and provides an invariant two-parametric generalization of Jones polynomial. Still three more steps need to be made for this. The first of them will extend Jones polynomial even further (sacrificing topological invariance), but reveal a new hidden structure of auxiliary vector spaces at the hypercube vertices. Then, after a kind of a supersymmetrization, we return back to topological invariant quantities – with a new deformation parameter gained in the process.

3 step: Cobordisms, cut-and-join operator and *extended* Jones

Resolution r of the graph Γ substitutes it by a set of disconnected cycles γ_a , $a \in r$, where a is some label, used to enumerate all cycles, arising in all possible resolutions. Each cycle had a length n_a (consists of n_a edges),

and there is an obvious constraint

$$\sum_{a \in r} n_a = n \quad \forall \text{ resolution } r \text{ of } \Gamma \quad (15)$$

Also $\gamma_a \cap \gamma_b = \emptyset$, i.e. the two cycles do not contain common edges, if γ_a and γ_b belong to the same resolution r , $a, b \in r$. At the same time, particular γ_a can participate in different resolutions r of Γ .

Elementary flips $\left| \right| \longleftrightarrow \left| \right|$ act on the set of cycles $\{\gamma_a\}$, either gluing two into one or splitting one into two:

$$\gamma_a = \gamma_b \frown \gamma_c, \quad n_a = n_b + n_c \quad (16)$$

and we introduce the cobordism structure constant N_{bc}^a , which is equal to one, if a flip exists, connecting the triple $(a|bc)$, and zero otherwise. By this definition, $N_{bc}^a = N_{cb}^a$, but $N_{bc}^a = 0$ for $b = c$. There are "undecomposable" cycles γ_a , for which all $N_{bc}^a = 0$, and a sequence of flips can be used to decompose arbitrary flip into such "minimal" ones – but we do not use this additional structure in the present paper.

In order to develop a more algebraic formalism, we associate with each cycle γ_a a "time-variable" p_a . Conceptually, it can be interpreted as a Wilson-loop variable $p_a = \text{tr } U^{n_a}$ with some matrix abstract matrix U , defined on the edges of Γ (in the simplest case, just constant) – but again we do not follow this line in the present text. Instead we proceed to the hypercube. With each hypercube vertex, i.e. with every resolution r we can now associate a monomial $p^r \equiv \prod_{a \in r} p_a$ as a product of all the $\nu(r)$ time-variables, associated with cycles which participate in the resolution r . Edges of the hypercube correspond to the elementary flips between "adjacent" resolutions,³ and each such flip is associated with exactly one item in the *cut-and-join operator*

$$W^\Gamma = \frac{1}{2} \sum_{a,b,c} N_{bc}^a \left(p_a \frac{\partial^2}{\partial p_b \partial p_c} + p_b p_c \frac{\partial}{\partial p_a} \right) \quad (17)$$

(note that there are no factors like $n_b n_c$ or $n_a = n_b + n_c$ in this formula, as in more familiar cut-and-join operators of [45] – instead there can be numerous p_a with the same a).

As before, the coloring of the vertices in the graph Γ_c defines the "starting" vertex r_c of the hypercube and all other vertices acquire the numbers $|r - r_c|$, equal to their distance from r_c (i.e. the number of edges, connecting them to r_c). Also all edges of the hypercube turn into arrows, pointing away from r_c – and for each triple $(a|bc)$, contributing to (17) one of the two terms is selected from the cut-and-join operator.

The last thing to do at this step is to introduce the "extended Jones polynomial" (compare with extended HOMFLY of [43]):

$$\mathcal{J}^{\Gamma_c} \{t | p_k^a\} = \sum_{r \in \text{hypercube}} t^{|r - r_c|} \prod_{a \in r}^{ \nu(r) } p_a \quad (18)$$

It lives on the infinite-dimensional space of time-variables p_a , and can be a subject of character calculus and can be analyzed from the perspectives of integrability (τ -functions) theory. It, however, is not a real topological invariant: the topological invariant Jones polynomial arises after restriction to *topological locus*

$$p_a = D = q + q^{-1} \quad (19)$$

and for $t = -q$:

$$J^{\Gamma_c}(q) = q^{n \bullet} (-1/q^2)^{n \circ} \mathcal{J}^{\Gamma_c} \left\{ t = -q \middle| p_a = D \right\} \quad (20)$$

Before proceeding further, we consider examples – the same as in s.2.

Example: $n = 1$

³The fact that there are exactly n edges, terminating in each vertex and, more generally, that edges form a hypercube, are strong restrictions on the set of the structure constants $\{N_{bc}^a\}$, which are automatically satisfied, if we build operator W , starting from some graph Γ .

In this case we have just a single cycle of length 2 and two cycles of length 1. This time the hypercube is a segment with two vertices and $p^{[0]} = p_1 p'_1$, $p^{[1]} = p_2$. Depending on the color of the vertex of original graph we get two extended Jones polynomials

$$\mathcal{J}^\bullet = p_1 p'_1 + t p_2 \quad (21)$$

and

$$\mathcal{J}^\circ = p_2 + t p_1 p'_1 \quad (22)$$

interchanged by the action of cut-and-join operator

$$\hat{W} = p_2 \frac{\partial^2}{\partial p_1 \partial p'_1} + p_1^2 \frac{\partial}{\partial p_2} \quad (23)$$

$$\hat{W} \mathcal{J}^\bullet = \mathcal{J}^\circ, \quad \hat{W} \mathcal{J}^\circ = \mathcal{J}^\bullet \quad (24)$$

The ordinary unreduced Jones polynomials appear according to the rule (20):

$$\begin{aligned} \mathcal{J}^\bullet(q) &= q(D^2 - qD) = D, \\ \mathcal{J}^\circ(q) &= (-1/q^2)(D - qD^2) = D \end{aligned} \quad (25)$$

Example: $n = 2$

This time we have two variable p_4, p'_4 and four variables $\bar{p}_2, \bar{p}'_2, p_2, p'_2$ with decomposition rules encoded in the operator

$$\hat{W} = (p_4 + p'_4) \left(\frac{\partial^2}{\partial \bar{p}_2 \partial \bar{p}'_2} + \frac{\partial^2}{\partial p_2 \partial p'_2} \right) + (\bar{p}_2 \bar{p}'_2 + p_2 p'_2) \left(\frac{\partial}{\partial p_4} + \frac{\partial}{\partial p'_4} \right) \quad (26)$$

The four extended Jones polynomials

$$\begin{aligned} \mathcal{J}^{\bullet\bullet} &= \bar{p}_2 \bar{p}'_2 + t(p_4 + p'_4) + t^2 p_2 p'_2, \\ \mathcal{J}^{\bullet\circ} &= p_4 + t(\bar{p}_2 \bar{p}'_2 + p_2 p'_2) + t^2 p'_4, \\ \mathcal{J}^{\circ\bullet} &= p'_4 + t(\bar{p}_2 \bar{p}'_2 + p_2 p'_2) + t^2 p_4, \\ \mathcal{J}^{\circ\circ} &= p_2 p'_2 + t(p_4 + p'_4) + t^2 \bar{p}_2 \bar{p}'_2 \end{aligned} \quad (27)$$

are again intertwined by \hat{W} :

$$\begin{aligned} \hat{W} \mathcal{J}^{\bullet\bullet} &= \mathcal{J}^{\bullet\circ} + \mathcal{J}^{\circ\bullet} = \hat{W} \mathcal{J}^{\circ\circ}, \\ \hat{W} \mathcal{J}^{\bullet\circ} &= \mathcal{J}^{\bullet\bullet} + \mathcal{J}^{\circ\circ} = \hat{W} \mathcal{J}^{\circ\bullet} \end{aligned} \quad (28)$$

Example: $n = 3$, trefoil

The set of cycles consists of three p_6, p'_6, p''_6 , three p_4, p'_4, p''_4 , two p_3, p'_3 and three p_2, p'_2, p''_2 , related through

$$\begin{aligned} \hat{W} &= p_6 \left(\frac{\partial^2}{\partial p'_4 \partial p'_2} + \frac{\partial^2}{\partial p''_4 \partial p''_2} \right) + p'_6 \left(\frac{\partial^2}{\partial p_4 \partial p_2} + \frac{\partial^2}{\partial p''_4 \partial p''_2} \right) + p''_6 \left(\frac{\partial^2}{\partial p_4 \partial p_2} + \frac{\partial^2}{\partial p'_4 \partial p'_2} \right) + \\ &+ (p_6 + p'_6 + p''_6) \frac{\partial^2}{\partial p_3 \partial p'_3} + \left(p_4 \frac{\partial^2}{\partial p'_2 \partial p''_2} + p'_4 \frac{\partial^2}{\partial p_2 \partial p''_2} + p''_4 \frac{\partial^2}{\partial p_2 \partial p'_2} \right) + \\ &+ \left((p'_4 p'_2 + p''_4 p''_2) \frac{\partial}{\partial p_6} + (p_4 p_2 + p''_4 p''_2) \frac{\partial}{\partial p'_6} + (p_4 p_2 + p'_4 p'_2) \frac{\partial}{\partial p''_6} \right) + \\ &+ p_3 p'_3 \left(\frac{\partial}{\partial p_6} + \frac{\partial}{\partial p'_6} + \frac{\partial}{\partial p''_6} \right) + \left(p'_2 p''_2 \frac{\partial}{\partial p_4} + p_2 p''_2 \frac{\partial}{\partial p'_4} + p_2 p'_2 \frac{\partial}{\partial p''_4} \right) \end{aligned} \quad (29)$$

The corresponding extended Jones polynomials are:

$$\begin{aligned}
\mathcal{J}^{\bullet\bullet\bullet} &= p_3 p'_3 + t(p_6 + p'_6 + p''_6) + t^2(p_4 p_2 + p'_4 p'_2 + p''_4 p''_2) + t^3 p_2 p'_2 p''_2, \\
\mathcal{J}^{\circ\bullet\bullet} &= p_6 + t(p'_4 p'_2 + p''_4 p''_2 + p_3 p'_3) + t^2(p'_6 + p''_6 + p_2 p'_2 p''_2) + t^3 p_4 p_2, \\
\mathcal{J}^{\bullet\circ\bullet} &= p'_6 + t(p_4 p_2 + p'_4 p'_2 + p_3 p'_3) + t^2(p_6 + p'_6 + p_2 p'_2 p''_2) + t^3 p'_4 p'_2, \\
\mathcal{J}^{\bullet\bullet\circ} &= p''_6 + t(p_4 p_2 + p'_4 p'_2 + p_3 p'_3) + t^2(p_6 + p'_6 + p_2 p'_2 p''_2) + t^3 p''_4 p''_2, \\
\mathcal{J}^{\circ\circ\bullet} &= p_4 p_2 + t(p'_6 + p''_6 + p_2 p'_2 p''_2) + t^2(p'_4 p'_2 + p''_4 p''_2 + p_3 p'_3) + t^3 p_6, \\
\mathcal{J}^{\circ\bullet\circ} &= p'_4 p'_2 + t(p_6 + p'_6 + p_2 p'_2 p''_2) + t^2(p_4 p_2 + p'_4 p'_2 + p_3 p'_3) + t^3 p'_6, \\
\mathcal{J}^{\bullet\circ\circ} &= p''_4 p''_2 + t(p_6 + p'_6 + p_2 p'_2 p''_2) + t^2(p_4 p_2 + p'_4 p'_2 + p_3 p'_3) + t^3 p''_6, \\
\mathcal{J}^{\circ\circ\circ} &= p_2 p'_2 p''_2 + t(p_4 p_2 + p'_4 p'_2 + p''_4 p''_2) + t^2(p_6 + p'_6 + p''_6) + t^3 p_3 p'_3
\end{aligned} \tag{30}$$

and cut-and-join operator acts between them as follows:

$$\begin{aligned}
\hat{W} \mathcal{J}^{\bullet\bullet\bullet} &= \mathcal{J}^{\circ\bullet\bullet} + \mathcal{J}^{\bullet\circ\bullet} + \mathcal{J}^{\bullet\bullet\circ}, \\
\hat{W} \mathcal{J}^{\circ\bullet\bullet} &= \mathcal{J}^{\bullet\bullet\bullet} + \mathcal{J}^{\circ\circ\bullet} + \mathcal{J}^{\circ\bullet\circ}, \\
&\dots
\end{aligned} \tag{31}$$

Clearly, the action of \hat{W} on \mathcal{J}^{Γ_c} changes the colors of vertices in the graph:

$$\hat{W} \mathcal{J}^{\Gamma_c} = \sum_{c': |r_{c'} - r_c| = 1} \mathcal{J}^{\Gamma_{c'}} \tag{32}$$

and the sum is over all starting vertices in the hypercube, adjacent to r_c .

4 step. Towards quiver: from resolutions to vector spaces

The rule (19) is in fact very suggestive: usually [8, 43] topological locus is associated with putting $X = I$ in the Miwa transform $p_a = \text{tr } X^{n_a}$. Perhaps, surprisingly, (19) does not depend on n_a – this remains to be understood, but somehow n_a does not seem to play any role in the topological cobordism construction, underlying Khovanov's approach. Anyhow, what is beyond doubt, is the re-appearance of the D -dimensional (or q -graded 2-dimensional, if one prefers) vector space: it was actually present at the very beginning, at step 1, disappeared at steps 2 and 3 and is now back through the Miwa transform.

To be precise, instead of $\prod_{a \in r} p_a$ at every vertex of the hypercube we now put a tensor product of vector spaces $\otimes_{a \in r}^{\nu(r)} V_a$, all V_a isomorphic to the q -graded two-dimensional space V , $V_a \cong V$. Now we can rewrite (13) as

$$J_{\square}^{\Gamma_c}(q) = (-)^{n_{\circ}} q^{n_{\bullet} - 2n_{\circ}} \sum_r^{2^n} (-q)^{|r - r_c|} \dim_q(V^{\otimes \nu(r)}) = (-)^{n_{\circ}} q^{n_{\bullet} - 2n_{\circ}} \sum_{I=0}^n (-q)^I \dim_q C_I \tag{33}$$

where the new vector spaces C_I are direct sums of all the vector spaces, associated with all the hypercube vertices, lying at a given distance from initial vertex r_0 :

$$C_I = \oplus_{r: |r - r_c| = I} V^{\otimes \nu(r)} \tag{34}$$

To define quantum dimension it is enough to fix a basis in V so that the two basis vectors v_{\pm} have weights $q^{\pm 1}$.

This ends Kauffman's reformulation of the ordinary Jones polynomial, and brings it to the form, allowing Khovanov's deformation. In original formulation topological invariance followed immediately from the obvious locality of the contracted product of \mathcal{R} -matrices and from the properties of Kauffman's \mathcal{R} -matrices (behavior under three Reidemeister moves). In (33) locality is less transparent. The idea now is that topological information is now encoded in reshuffling properties of the graph under the elementary flips of resolutions at particular vertices. While flips are local, reshuffling s are not. By adding a new vertex, like in the Reidemeister move R_1 we add a new dimension to the hypercube, and locality should be now treated in homological terms. But for this to work one needs to reformulate (33) in terms of cohomologies.

For this we should recall that there are operators acting along the hypercube edges and lift them to operators, acting between tensor products of vector spaces V . After that – if we are lucky – the linear combinations of these operators, arising in projection to the spaces C_I , will provide *differentials* (possess the property $d_{i+1} d_I = 0$), allowing a cohomological reformulation of (33).

It turns out that the differentials are indeed produced by a kind of supersymmetrization of the cut-and-join operator (17), but before we switch to this issue (step 6), we describe what we need it for.

5 step. Restricting to kernels

Eq.(33) already looks like an Euler characteristic of the graded complex

$$\mathcal{C} : \quad 0 \longrightarrow C_0 \xrightarrow{d_1} C_1 \xrightarrow{d_2} \dots \xrightarrow{d_n} C_n \longrightarrow 0 \quad (35)$$

and Euler characteristic can be alternatively rewritten in terms of cohomologies:

$$\begin{aligned} J_{\square}^{\Gamma_c}(q) &= (-)^{n_{\circ}} q^{n_{\bullet} - 2n_{\circ}} \sum_{I=0}^n (-q)^I \dim_q C_I = (-)^{n_{\circ}} q^{n_{\bullet} - 2n_{\circ}} \sum_{I=0}^n (-q)^I \dim_q H_I = \\ &= \sum_{I=0}^n (-q)^I \left\{ \dim_q \left(\text{Ker}(d_{I+1}) \right) - \dim_q \left(\text{Im}(d_I) \right) \right\} \end{aligned} \quad (36)$$

where at the last stage all the q -factors are absorbed into regrading of the C_I spaces.

This formula already allows a T -deformation: the Jones superpolynomial

$$\boxed{P_{\square}^{\Gamma_c}(T, q) = q^{n_{\bullet}} \cdot (Tq^2)^{-n_{\circ}} \sum_{I=0}^n (qT)^I \left\{ \dim_q \left(\text{Ker}(d_{I+1}) \right) - \dim_q \left(\text{Im}(d_I) \right) \right\}} \quad (37)$$

is invariant under Reidemeister moves, provided the differentials d_I are chosen in appropriate way.

Again, invariance is a two-level statement:

(i) the Poincare polynomial (37) is *local* in the sense that it is not changed if the complex \mathcal{C} is substituted by \mathcal{C}' such that either the factor-complex \mathcal{C}/\mathcal{C}' is acyclic (has vanishing cohomologies) or $\mathcal{C}' = \mathcal{C}/\mathcal{C}''$ with acyclic \mathcal{C}'' ,

(ii) differentials d_I are such, that the changes of complex \mathcal{C} , associated with the Reidemeister moves, provide acyclic factor-complexes.

Reverting colors of all vertices, $\Gamma_c \longrightarrow \Gamma_{\bar{c}}$, reverses orientation of the link, and the superpolynomial changes in a simple way:

$$P_{\square}^{\Gamma_{\bar{c}}}(T, q) = P_{\square}^{\Gamma_c}(T^{-1}, q^{-1}) \quad (38)$$

(in colored superpolynomials this Z_2 mirror-like transform also transposes the Young diagram, which describes the representation).

Differentials d_i are linear combinations (actually, sums with plus and minus signs) of elementary linear maps, associated with particular edges of the hypercube. We already now the operators, associated with elementary flips in the cut-and-join operator (17). Now we should promote them to linear maps, acting between the tensor products of vector spaces V .

6 step. Specifying the differentials

6.1 The basic maps

Since every elementary map is associated with some flip of the resolutions, which splits one of the cycles γ_a into two $\gamma_b \approx \gamma_c$ or glues two into one, the map should act either as $Q : V_a \longrightarrow V_b \otimes V_c$ or as $Q^* : V_b \otimes V_c \longrightarrow V_a$ respectively. Choosing the q -graded basis v_{\pm} in each 2-dimensional V , we can describe the maps again in the form of a cut-and-join operator

$$\hat{w} = \underbrace{(v_+ \otimes v_- + v_- \otimes v_+) \frac{\partial}{\partial v_+} + v_- \otimes v_- \frac{\partial}{\partial v_-}}_Q + \underbrace{v_+ \frac{\partial}{\partial v_+} \otimes \frac{\partial}{\partial v_+} + v_- \left(\frac{\partial}{\partial v_+} \otimes \frac{\partial}{\partial v_-} + \frac{\partial}{\partial v_-} \otimes \frac{\partial}{\partial v_+} \right)}_{Q^*} \quad (39)$$

The two constituents Q and Q^* of W can be represented as 4×2 and 2×4 matrices respectively:

$$\begin{aligned} Q : \quad V &\longrightarrow V \otimes V & \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} &= \begin{pmatrix} Q_{++}^+ & Q_{++}^- \\ Q_{+-}^+ & Q_{+-}^- \\ Q_{-+}^+ & Q_{-+}^- \\ Q_{--}^+ & Q_{--}^- \end{pmatrix} \\ Q^* : \quad V \otimes V &\longrightarrow V & \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix} &= \begin{pmatrix} Q_{++}^{++} & Q_{++}^{+-} & Q_{++}^{-+} & Q_{++}^{--} \\ Q_{+-}^{++} & Q_{+-}^{+-} & Q_{+-}^{-+} & Q_{+-}^{--} \\ Q_{-+}^{++} & Q_{-+}^{+-} & Q_{-+}^{-+} & Q_{-+}^{--} \\ Q_{--}^{++} & Q_{--}^{+-} & Q_{--}^{-+} & Q_{--}^{--} \end{pmatrix} \end{aligned} \quad (40)$$

Tensorial notation Q_{ij}^k and Q_i^{jk} for Q and Q^* respectively are useful for many purposes. In particular, in these terms it is easy to formulate important properties of Q^* and Q : they can be considered as defining commutative

$$\boxed{Q_{jk}^i = Q_{kj}^i, \quad Q_k^{ij} = Q_k^{ji}} \quad (41)$$

and associative

$$\boxed{[\check{Q}_j, \check{Q}_k] = 0, \quad [\check{Q}^j, \check{Q}^k] = 0} \quad (42)$$

algebra. Here the linear maps \check{Q}_i and \check{Q}^i are defined as $(\check{Q}^i)_k^j = Q_k^{ij}$ and $(\check{Q}_i)^j_k = Q_{ik}^j$. The last property is trivial, because $\check{Q}^- = \check{Q}_+ = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ are unit matrices. The other two also coincide: $\check{Q}^+ = \check{Q}_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$.

The shape of operations Q^* and Q is severely restricted by the requirement that they respect q -gradation $\text{grad}_q(v^\pm) = \pm 1$ – in fact they diminish it by one:

$$\begin{aligned} Q^*(v^- \otimes v^-) &= 0, & q^{-1} \cdot q^{-1} &\longrightarrow 0 \\ Q^*(v^+ \otimes v^-) &= Q^*(v^- \otimes v^+) = v^-, & q^{-1} \cdot q &\longrightarrow q^{-1} \\ Q^*(v^+ \otimes v^+) &= v^+, & q \cdot q &\longrightarrow q \\ Q(v^+) &= v^+ \otimes v^- + v^- \otimes v^+, & q &\longrightarrow q^{-1} \cdot q \\ Q(v^-) &= v^- \otimes v^-, & q^{-1} &\longrightarrow q^{-1} \cdot q^{-1} \end{aligned} \quad (43)$$

In fact, the factor $q^{|r-r_c|}$, which is explicitly introduced into the definition of Jones polynomials, is needed to compensate this gradation drop.

6.2 Differentials and auxiliary Koszul complex

As usual, from commuting matrices \tilde{Q}^i , which satisfy (41), we can construct Koszul-like differentials, acting on the functions of anticommuting variables ϑ_i^a , $\vartheta_i^a \vartheta_j^b + \vartheta_j^b \vartheta_i^a = 0$ with arbitrary set of indices $\{a\}$ (see [46] for a review in the general context of *non-linear algebra* [47]). We take for this indices the labels of the cycles γ_a and form a Koszul counterpart of the cut-and-join operator (17):

$$\Omega^\Gamma = \sum_{\substack{a \\ b < c}} \epsilon_{bc}^a N_{bc}^a \sum_{i,j,k} \left(Q_k^{ij} \vartheta_a^k \frac{\partial^2}{\partial \vartheta_b^i \partial \vartheta_c^j} + Q_{ij}^k \vartheta_b^i \vartheta_c^j \frac{\partial}{\partial \vartheta_a^k} \right) \quad (44)$$

The ϑ product is antisymmetric in bc , therefore to count each pair bc once one now sums over $b < c$ instead of dividing by two. We remind that by definition $N_{bc}^a = 0$ for $b = c$. The sign factor $\epsilon_{bc}^a = \pm$ can be chosen in different ways. The choice, suggested in [22], uses the fact that every item with $N_{bc}^a \neq 0$ is associated with an edge of a hypercube, which connects two hypercube vertices, differing by just one digit \star in the notation $r = [\alpha_1 \dots \alpha_{m-1}, \star, \alpha_{m+1} \dots \alpha_n]$. If \star stands at the m -th place, then

$$\epsilon_{bc}^a = (-)^{\alpha_1 + \dots + \alpha_{m-1}} \quad (45)$$

As in (17) one can associate particular items in this operator with linear maps, acting along the edges of the hypercube, and the choice between the two terms in the sum is dictated by direction of the arrow along the edge.

Operator Ω is not nilpotent $\Omega^2 \neq 0$, it is rather a sum of a BRST-like operator and its conjugate: $\Omega = Q + \tilde{Q}$. The nilpotent piece, Q , can be decomposed into particular pieces, $Q = \sum_I d_I$ with the property $d_{I+1} d_I = 0$, and these d_I can be used as differentials of the complex \mathcal{C} . Moreover, different colorings Γ_c of the graph Γ are associated with different complexes and with different decompositions of $\Omega = Q + \tilde{Q}$. The sum of differentials d_I for every Γ_c contains exactly one half of the items of Ω – since arrows along all the edges of hypercube point in a definite direction.

Decomposition, associated with Γ^c can be easily found, looking at *extended* Jones polynomial \mathcal{J}^{Γ^c} . To find d_I one picks up the items of cut-and-join operator \tilde{W}^Γ in (17), which convert the terms with t^{I-1} in \mathcal{J}^{Γ^c} into those with t^I . Then the same terms in Q form the relevant d_I . In the remaining part of this section we illustrate this rule by several examples.

From now on we use the condensed notation, suppressing the tensor Q :

$$\begin{aligned}\hat{Q}_{bc}^{a\downarrow} &= \vartheta_b \vartheta_c \frac{\partial}{\partial \vartheta_a} \equiv Q_{ij}^k \vartheta_b^i \vartheta_c^j \frac{\partial}{\partial \vartheta_a^k} = \theta_b \theta_c \frac{\partial}{\partial \theta_a} + (\theta_b \eta_c + \eta_c \theta_b) \frac{\partial}{\partial \eta_a}, \\ \hat{Q}_{a\uparrow}^{bc} &= \vartheta_a \frac{\partial^2}{\partial \vartheta_b \partial \vartheta_c} \equiv Q_k^{ij} \vartheta_a^k \frac{\partial^2}{\partial \vartheta_b^i \partial \vartheta_c^j} = \theta_a \left(\frac{\partial^2}{\partial \eta_b \partial \theta_c} + \frac{\partial^2}{\partial \theta_b \partial \eta_c} \right) + \eta_a \frac{\partial^2}{\partial \eta_b \partial \eta_c}\end{aligned}\quad (46)$$

where at the r.h.s. we also denoted $\vartheta_a^+ = \eta_a$ and $\vartheta_a^- = \theta_a$. The properties (41) and (42) of Q allow to perform cyclic permutations of ϑ – and this is sufficient to check the properties $d_{I+1} d_I$ of the differentials in all the examples below. Also in what follows the indices i, j, k will be used for other purposes, and no longer take values \pm .

6.3 Decomposition into differentials d_I for particular Γ_c

It can deserve repeating the whole construction of d_I once again.

Cut-and-join operator is associated with the graph Γ (the colors of vertices forgotten):

$$\hat{W}^\Gamma = \sum_{E \in \text{hypercube}^\Gamma} \sum_{\beta = \uparrow, \downarrow} \hat{w}_E^{(\beta)} \quad (47)$$

where the sum is over edges E of the hypercube associated with Γ , and over their orientations β . Elementary operators are of the form $\hat{w}_E^{(\downarrow)} = p^2 \frac{\partial}{\partial p}$ and $\hat{w}_E^{(\uparrow)} = p \frac{\partial^2}{\partial p^2}$. If Γ has n vertices, then the 2^n vertices of the hypercube are labeled by binary numbers $[\alpha_1 \dots \alpha_n]$ and its $2^{n-1}n$ edges – by $[\alpha_1 \dots \alpha_{m-1} \star \alpha_{m+1} \dots \alpha_m]$: this edge connects vertices $[\alpha_1 \dots \alpha_{m-1}, 0, \alpha_{m+1} \dots \alpha_m]$ and $[\alpha_1 \dots \alpha_{m-1}, 1, \alpha_{m+1} \dots \alpha_m]$.

With edge E we associate a sign factor

$$\epsilon_E = (-)^{\alpha_1 + \dots + \alpha_{m-1}} \quad (48)$$

Each elementary operator $\hat{w}_E^{(\beta)}$ has a "supersymmetric" counterpart, see (46). Thus one can construct

$$\mathfrak{Q}^\Gamma = \sum_E \sum_{\beta = \uparrow, \downarrow} \epsilon_E \hat{Q}_E^{(\beta)} \quad (49)$$

If coloring Γ_c of Γ is fixed, this implies a choice of orientation β_E^c for each edge. This turns an edge into an arrow, which has a *tail*: the first of the two vertices, that it connects. The choice of β_E^c splits \mathfrak{Q} into two "conjugate" halves, we pick one of them, which already has chances to be nilpotent:

$$\begin{aligned}\mathfrak{Q}^\Gamma &= \mathcal{Q}^{\Gamma_c} + \widetilde{\mathcal{Q}^{\Gamma_c}}, \\ \mathcal{Q}^{\Gamma_c} &= \sum_E \epsilon_E \hat{Q}_E^{(\beta_E^c)}\end{aligned}\quad (50)$$

Existence of *tails* allows to further split it into n items:

$$\begin{aligned}\mathcal{Q}^{\Gamma_c} &= \sum_{I=0}^n d_I, \\ \boxed{d_I^{\Gamma_c} = \sum_{E: |\text{tail}_E - r_c| = I} \epsilon_E \hat{Q}_E^{(\beta_E^c)}}\end{aligned}\quad (51)$$

These $d_I^{\Gamma_c}$ are the differentials, associated with Γ_c .

However, as we already mentioned, from practical point of view it is often more convenient to simply read the set of items (51) from the expression for extended Jones polynomial.

6.4 The case of $q = 1$: The polynomial $p(T) = P(T, q)|_{q=1}$

When $q = 1$, it is enough to calculate the ranks of the matrices d_I . All the rest follows from linear algebra relations:

$$\text{Rank} = \dim(\text{Coker}) \quad (52)$$

and therefore

$$\begin{aligned}\dim(\text{Ker}(d_I)) &= \dim(C_{I-1}) - \dim(\text{Coker}(d_I)) = \underline{\dim(C_{I-1}) - \text{Rank}(d_I)} = \dim(\text{Corank}(d_I)), \\ \dim(\text{Im}(d_I)) &= \dim(\text{Coker}(\tilde{d}_I)) = \text{Rank}(\tilde{d}_I) = \underline{\text{Rank}(d_I)} = \dim(C_I) - \dim(\text{Corank}(\tilde{d}_I)), \\ \dim(\text{Ker}(d_I)) + \dim(\text{Im}(d_I)) &= \dim(C_{I-1})\end{aligned}\quad (53)$$

so that

$$p(T) = \sum_{I=0}^n T^{I-n_\circ} \left\{ \dim(\text{Ker}(d_{I+1})) - \dim(\text{Im}(d_I)) \right\} = \sum_{I=0}^n T^{I-n_\circ} \left\{ \dim(C_I) - \text{Rank}(d_{I+1}) - \text{Rank}(d_I) \right\} \quad (54)$$

where for the two boundary operators we have $\text{Rank}(d_0) = \text{Rank}(d_{n+1}) = 0$.

In order to restore the q -dependence we need not just *dimensions* of the cohomologies, but q -dimensions, involving the q -grading numbers of all their constituents:

$$\begin{aligned}P(T, q) &= \frac{q^{n_\bullet}}{(q^2 T)^{n_\circ}} \sum_{I=0}^n (qT)^I \left\{ \dim_q(\text{Ker}(d_{I+1})) - \dim_q(\text{Im}(d_I)) \right\} = \\ &= \frac{q^{n_\bullet}}{(q^2 T)^{n_\circ}} \sum_{I=0}^n (qT)^I \left\{ \dim_q(C_I) - \text{Rank}_q(d_{I+1}) - q^{-1} \text{Rank}_q(d_I) \right\} = \\ &= \frac{q^{n_\bullet}}{(q^2 T)^{n_\circ}} \left(\sum_{I=0}^n (qT)^I \dim_q(C_I) - (1+T) \sum_{I=1}^n (qT)^{I-1} \text{Rank}_q(d_I) \right) = \\ &= \frac{q^{n_\bullet}}{(q^2 T)^{n_\circ}} \left((1+T) \sum_{I=1}^n (qT)^{I-1} \dim_q(\text{Ker}(d_I)) - T \sum_{I=0}^{n-1} (qT)^I \dim_q(C_I) + (qT)^n \dim_q(C_n) \right)\end{aligned}\quad (55)$$

In practice this means that the basis vectors of $H_I = \text{Ker}(d_{I+1})/\text{Im}(d_I)$ should be found explicitly. Sometime, technically more convenient is to look for the zero-modes of Laplace operators $\tilde{d}_{I+1}d_{I+1}$ and $d_I\tilde{d}_I$ perhaps, even among all their eigenvectors (in the framework of the present paper all vector spaces are finite-dimensional).

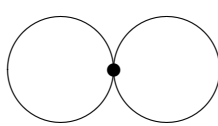
From the last two lines of (55) it is clear that for $T = -1$ the superpolynomial turns back to Jones (36)

$$P(T, q) \Big|_{T=-1} = J(q) \quad (56)$$

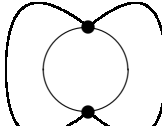
The q -Euler characteristic of a complex can be calculated both from q -dimensions of cohomologies and of entire spaces C_I .

7 step. Calculating cohomologies

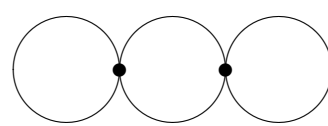
Now it remains to find the ranks, kernels and images of operators d_I . We continue with the example of 2-strand braids with n black vertices. We make just a minor, but important deviation in sec.7.3 to consider a very simple non-braid example.



eight
sec.7.1



Hopf
sec.7.2



double eight
sec.7.3

7.1 Example. An eight: two loops intersecting at one vertex ($n = 1$)

Extended Jones polynomials in this case are $\mathcal{J}^\bullet = p_1 p'_1 + t p_2$ and $\mathcal{J}^\circ = p_2 + t p_1 p'_1$. Thus the differentials in the two cases look like

$$d_1^\bullet = \vartheta_2 \frac{\partial^2}{\partial \vartheta_1 \partial \vartheta'_1} = \theta_2 \left(\frac{\partial^2}{\partial \eta_1 \partial \theta'_1} + \frac{\partial^2}{\partial \theta_1 \partial \eta'_1} \right) + \eta_2 \frac{\partial^2}{\partial \eta_1 \partial \eta'_1} = \hat{Q}_{2\uparrow}^{11'} \quad (57)$$

and

$$d_1^\circ = \vartheta_1 \vartheta'_1 \frac{\partial}{\partial \vartheta_2} = \theta_1 \theta'_1 \frac{\partial}{\partial \theta_2} + (\theta_1 \eta'_1 + \eta_1 \theta'_1) \frac{\partial}{\partial \eta_2} = \hat{Q}_{11'}^{2\downarrow} \quad (58)$$

i.e. exactly the operators from (46). These two operators are nilpotent and conjugate to each other, thus the square $\mathcal{Q} = d_1^\bullet + d_1^\circ$ is a non-vanishing "Hamiltonian".

Kernels of these operators depend very much on the space, where they act. If these spaces are arbitrary functions of Grassmannian variables $\theta_1, \theta'_1, \theta_2, \eta_1, \eta'_1, \eta_2$ the kernels are huge. However, we are interested in concrete spaces: 4-dimensional $V \otimes V$, spanned by bilinear functions of ϑ_1 and ϑ'_1 and 2-dimensional V spanned by homogeneous linear function of ϑ_2 .

7.1.1 The \bullet case

The map

$$d_1^\bullet : V \otimes V \longrightarrow V, \quad d_1^\bullet \left(\underbrace{\theta_1 \theta'_1}_{q^{-2}}, \theta_1 \eta'_1 + \eta_1 \theta'_1, \underbrace{\theta_1 \eta'_1 - \eta_1 \theta'_1}_{q^0}, \eta_1 \eta'_1 \right) = (0, -2\theta_2, 0, -\eta_2) \quad (59)$$

has rank two, 2-dimensional kernel and vanishing coimage. Thus

$$p^\bullet(T) \stackrel{(54)}{=} \left\{ \dim(C_0) - \text{Rank}(d_1) \right\} + T \left\{ \dim(C_1) - \text{Rank}(d_1) \right\} = (4 - 2) + T(2 - 2) = 2; \\ P^\bullet(T, q) = q \left(\dim_q(\text{Ker}(d_1)) \right) + qT \dim_q(\text{Coim}(d_1)) = q(q^{-2} + 1) = D \quad (60)$$

7.1.2 The \circ case

At the same time, the map

$$d_1^\circ : V \longrightarrow V \otimes V, \quad d_1^\circ(\theta_2, \eta_2) = (\theta_1 \theta'_1, (\theta_1 \eta'_1 + \eta_1 \theta'_1)) \quad (61)$$

also has rank two, but vanishing kernel and 2-dimensional coimage, with the basis $\left(\underbrace{(\theta_1 \eta'_1 + \eta_1 \theta'_1)}_{q^0}, \underbrace{\eta_1 \eta'_1}_{q^2} \right)$. Thus

$$p^\circ(T) \stackrel{(54)}{=} T^{-1} \left(\left\{ \dim(C_0) - \text{Rank}(d_1) \right\} + T \left\{ \dim(C_1) - \text{Rank}(d_1) \right\} \right) = (2 - 2) + T(4 - 2) = 2; \\ P^\circ(T, q) = \frac{1}{q^2 T} \left(\dim_q(\text{Ker}(d_1)) \right) + qT \dim_q(\text{Coim}(d_1)) = \frac{qT}{q^2 T} (q^2 + 1) = D \quad (62)$$

7.2 Example. Hopf link ($n = 2$)

In this case we have four different cycles of length 2 and two cycles of length 4. Extended Jones polynomials are

$$\begin{aligned} \mathcal{J}^{\bullet\bullet} &= \bar{p}_2 \bar{p}'_2 + t(p_4 + p'_4) + t^2 p_2 p'_2, \\ \mathcal{J}^{\bullet\circ} &= p_4 + t(\bar{p}_2 \bar{p}'_2 + p_2 p'_2) + t^2 p'_4, \\ \mathcal{J}^{\circ\bullet} &= p'_4 + t(\bar{p}_2 \bar{p}'_2 + p_2 p'_2) + t^2 p_4, \\ \mathcal{J}^{\circ\circ} &= p_2 p'_2 + t(p_4 + p'_4) + t^2 \bar{p}_2 \bar{p}'_2 \end{aligned} \quad (63)$$

7.2.1 The $\bullet\bullet$ case

This means that in the first case the differentials are:

$$\begin{aligned} d_1^{\bullet\bullet} &= (\vartheta_4 + \vartheta'_4) \frac{\partial^2}{\partial \vartheta_2 \partial \vartheta'_2} = (\theta_4 + \theta'_4) \left(\frac{\partial^2}{\partial \eta_2 \partial \theta'_2} + \frac{\partial^2}{\partial \theta_2 \partial \eta'_2} \right) + (\eta_4 + \eta'_4) \frac{\partial^2}{\partial \eta_2 \partial \eta'_2}, \\ d_2^{\bullet\bullet} &= \vartheta_2 \vartheta'_2 \left(\frac{\partial}{\partial \vartheta_4} - \frac{\partial}{\partial \vartheta'_4} \right) = \theta_2 \theta'_2 \left(\frac{\partial}{\partial \theta_4} - \frac{\partial}{\partial \theta'_4} \right) + (\theta_2 \eta'_2 + \eta_2 \theta'_2) \left(\frac{\partial}{\partial \eta_4} - \frac{\partial}{\partial \eta'_4} \right) \end{aligned} \quad (64)$$

$$\begin{aligned}
\text{Ker}(d_1^{\bullet\bullet}) &= \left\{ \theta_2 \theta'_2, \eta_2 \theta'_2 - \theta_2 \eta'_2 \right\}, \\
\text{Im}(d_1^{\bullet\bullet}) &= \left\{ \theta_4 + \theta'_4, \eta_4 + \eta'_4 \right\} = \text{Ker}(d_2^{\bullet\bullet}), \\
\text{Im}(d_2^{\bullet\bullet}) &= \left\{ \theta_2 \theta'_2, \theta_2 \eta'_2 + \eta_2 \theta'_2 \right\} \implies \text{Coim}(d_2^{\bullet\bullet}) = \left\{ \eta_2 \eta'_2, \theta_2 \eta'_2 - \eta_2 \theta'_2 \right\}
\end{aligned} \tag{65}$$

and

$$\begin{aligned}
p^{\bullet\bullet}(T) &\stackrel{(54)}{=} \left\{ \dim(C_0) - \text{Rank}(d_1) \right\} + T \left\{ \dim(C_1) - \text{Rank}(d_1) - \text{Rank}(d_2) \right\} + T^2 \left\{ \dim(C_2) - \text{Rank}(d_2) \right\} = \\
&= (2^2 - 2)T^0 + (2 + 2 - 2 - 2)T^1 + (2^2 - 2)T^2 = 2(1 + T^2),
\end{aligned} \tag{66}$$

$$\begin{aligned}
P^{\bullet\bullet}(T, q) &= q^2 \left(T^0 \dim_q(H_0^{\bullet\bullet}) + (qT)^1 \dim_q(H_1^{\bullet\bullet}) + (qT)^2 \dim_q(H_2^{\bullet\bullet}) \right) = \\
&= q^2 \left\{ T^0 \dim_q \text{Ker}(d_1^{\bullet\bullet}) + (qT)^1 \left(\dim_q \text{Ker}(d_2^{\bullet\bullet}) - \dim_q \text{Im}(d_1^{\bullet\bullet}) \right) + (qT)^2 \dim_q \text{Coim}(d_2^{\bullet\bullet}) \right\} = \\
&= q^2 \left\{ T^0 (q^{-2} + 1) + (qT)^1 \cdot 0 + (qT)^2 (q^2 + 1) \right\} = qD(1 + q^4 T^2)
\end{aligned} \tag{67}$$

7.2.2 The $\circ\circ$ case

In the particular case of Hopf link (of $n = 2$) the problem with two white vertices is literally the same as with two black ones, because there is an (accidental) symmetry $(\vartheta_2, \vartheta'_2) \leftrightarrow (\vartheta_2, \vartheta'_2)$. The only difference in this case when one switches from black to white is the substitution of overall factor q^2 by $(q^2 T)^{-2}$, so that

$$P^{\circ\circ}(T, q) = (q^2 T)^{-2} \left(q^{-2} P^{\bullet\bullet}(T, q) \right) = q^{-1} D(1 + q^{-4} T^{-2}) = P^{\bullet\bullet}(T^{-1}, q^{-1}) \tag{68}$$

All intermediate steps in this formula are special for $n = 2$, but the final relation (38) is universal. However, already for 2-strand braids with $n > 2$ the cohomology calculus at two sides of the equality will be essentially different, see s.7.4 below.

7.2.3 The $\bullet\circ$ case

The case of two vertices of different color is different already for $n = 2$.

$$\begin{aligned}
d_1^{\bullet\circ} &= (\vartheta_2 \vartheta'_2 + \bar{\vartheta}_2 \bar{\vartheta}'_2) \frac{\partial}{\partial \vartheta_4} = (\theta_2 \theta'_2 + \bar{\theta}_2 \bar{\theta}'_2) \frac{\partial}{\partial \theta_4} + (\theta_2 \eta'_2 + \eta_2 \theta'_2 + \bar{\theta}_2 \bar{\eta}'_2 + \bar{\eta}_2 \bar{\theta}'_2) \frac{\partial}{\partial \eta_4}, \\
d_2^{\bullet\circ} &= \vartheta'_4 \left(\frac{\partial^2}{\partial \vartheta_2 \partial \vartheta'_2} - \frac{\partial^2}{\partial \bar{\vartheta}_2 \partial \bar{\vartheta}'_2} \right) = \theta'_4 \left(\frac{\partial^2}{\partial \eta_2 \partial \theta'_2} + \frac{\partial^2}{\partial \theta_2 \partial \eta'_2} - \frac{\partial^2}{\partial \bar{\eta}_2 \partial \bar{\theta}'_2} - \frac{\partial^2}{\partial \bar{\theta}_2 \partial \bar{\eta}'_2} \right) + \eta'_4 \left(\frac{\partial^2}{\partial \eta_2 \partial \eta'_2} - \frac{\partial^2}{\partial \bar{\eta}_2 \partial \bar{\eta}'_2} \right)
\end{aligned} \tag{69}$$

This time

$$\begin{aligned}
\text{Ker}(d_1^{\bullet\circ}) &= \emptyset \implies \text{Corank}(d_1) = 0, \text{Rank}(d_1) = 2 - 0 = 2, \\
\text{Im}(d_1^{\bullet\circ}) &= \left\{ \theta_2 \theta'_2 + \bar{\theta}_2 \bar{\theta}'_2, \theta_2 \eta'_2 + \eta_2 \theta'_2 + \bar{\theta}_2 \bar{\eta}'_2 + \bar{\eta}_2 \bar{\theta}'_2 \right\}, \\
\text{Coker}(d_2^{\bullet\circ}) &= \left\{ \theta_2 \eta'_2 + \eta_2 \theta'_2 - \bar{\theta}_2 \bar{\eta}'_2 - \bar{\eta}_2 \bar{\theta}'_2, \eta_2 \eta'_2 - \bar{\eta}_2 \bar{\eta}'_2 \right\}, \\
\text{Im}(d_2^{\bullet\circ}) &= \left\{ \theta'_4, \eta'_4 \right\} \implies \text{Coim}(d_2^{\bullet\circ}) = \emptyset, \text{Rank}(d_2) = 2
\end{aligned} \tag{70}$$

so that

$$\begin{aligned}
p^{\bullet\circ}(T) &\stackrel{(54)}{=} T^{-1} \left(\left\{ \dim(C_0) - \text{Rank}(d_1) \right\} + T \left\{ \dim(C_1) - \text{Rank}(d_1) - \text{Rank}(d_2) \right\} + T^2 \left\{ \dim(C_2) - \text{Rank}(d_2) \right\} \right) = \\
&= T^{-1} \left((2 - (2 - 0))T^0 + (2^2 + 2^2 - 2 - 2)T^1 + (2 - 2)T^2 \right) = 4,
\end{aligned} \tag{71}$$

$$\begin{aligned}
P^{\bullet\circ}(T, q) &= \frac{q}{q^2 T} \left(T^0 \dim_q(H_0^{\bullet\circ}) + (qT)^1 \dim_q(H_1^{\bullet\circ}) + (qT)^2 \dim_q(H_2^{\bullet\circ}) \right) = \\
&= \frac{q}{q^2 T} \left\{ T^0 \dim_q \text{Ker}(d_1^{\bullet\circ}) + (qT)^1 \left(\dim_q \text{Ker}(d_2^{\bullet\circ}) - \dim_q \text{Im}(d_1^{\bullet\circ}) \right) + (qT)^2 \dim_q \text{Coim}(d_2^{\bullet\circ}) \right\} = \\
&= (qT)^{-1} \left(T^0 \cdot 0 + (qT)^1 \left((2D^2 - q^2 - 1) - (q^{-2} + 1) \right) + (qT)^2 \cdot 0 \right) = D^2 = \left(P^{\text{uknot}}(q, T) \right)^2
\end{aligned} \tag{72}$$

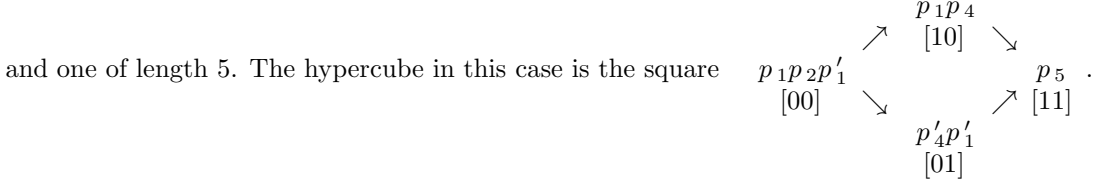
as it should be.

The case $\bullet\bullet$ is absolutely the same.

7.3 Example. Double eight ($n = 2$)

In fact for $n = 2$ there is another knot diagram: "double eight", which is not a braid. This knot is topologically trivial for all colorings of the two vertices, still its consideration is quite instructive.

Four double eight we have four cycles: two complementary pairs of lengths 1 and 4, one cycle of length 2



7.3.1 The $\bullet\bullet$ case

The choice of arrows is shown for the case two black vertices, when

$$\mathcal{J}^{\bullet\bullet} = p_1 p_2 p'_1 + t(p_1 p_4 + p'_1 p'_4) + t^2 p_5 \quad (73)$$

The two differentials are

$$\begin{aligned}
 d_1^{\bullet\bullet} &= \vartheta_4 \frac{\partial^2}{\partial \vartheta_2 \partial \vartheta'_1} + \vartheta'_4 \frac{\partial^2}{\partial \vartheta_2 \partial \vartheta_1} = \theta_4 \left(\frac{\partial^2}{\partial \eta_2 \partial \theta'_1} + \frac{\partial^2}{\partial \theta_2 \partial \eta'_1} \right) + \eta_4 \frac{\partial^2}{\partial \eta_2 \partial \theta'_1} + \theta'_4 \left(\frac{\partial^2}{\partial \eta_2 \partial \theta_1} + \frac{\partial^2}{\partial \theta_2 \partial \eta_1} \right) + \eta'_4 \frac{\partial^2}{\partial \eta_2 \partial \eta_1}, \\
 d_2^{\bullet\bullet} &= \vartheta_5 \left(\frac{\partial^2}{\partial \vartheta_1 \partial \vartheta_4} + \frac{\partial^2}{\partial \vartheta'_1 \partial \vartheta'_4} \right) = \theta_5 \left(\frac{\partial^2}{\partial \eta_1 \partial \theta_4} + \frac{\partial^2}{\partial \theta_1 \partial \eta_4} + \frac{\partial^2}{\partial \eta'_1 \partial \theta'_4} + \frac{\partial^2}{\partial \theta'_1 \partial \eta'_4} \right) + \eta_5 \left(\frac{\partial^2}{\partial \eta_1 \partial \eta_4} + \frac{\partial^2}{\partial \eta'_1 \partial \eta'_4} \right) \quad (74)
 \end{aligned}$$

Clearly, $d_2^{\bullet\bullet} d_1^{\bullet\bullet} = 0$ and

$$\begin{aligned}
 \text{Ker}(d_1^{\bullet\bullet}) &= \left\{ \theta_1 \theta_2 \theta'_1, \theta_1 \theta_2 \eta'_1 - \theta_1 \eta_2 \theta'_1 + \eta_1 \theta_2 \theta'_1 \right\}, \quad \dim_q \text{Ker}(d_1) = q^{-3} + q^{-1}, \\
 \text{Coim}(d_1^{\bullet\bullet}) &= \text{Coker}(d_2^{\bullet\bullet}) = \left\{ \eta_1 \theta_4 + \theta_1 \eta_4 + \eta'_1 \theta'_4 + \theta'_1 \eta'_4, \eta_1 \eta_4 + \eta'_1 \eta'_4 \right\}, \\
 \text{Im}(d_2^{\bullet\bullet}) &= \left\{ \theta_5, \eta_5 \right\}, \quad \text{Coim}(d_2^{\bullet\bullet}) = \emptyset \quad (75)
 \end{aligned}$$

At $q = 1$, since

$$\dim(C_0) = \dim(C_1) = 8, \quad \dim(C_2) = 2 \quad \text{and} \quad \dim(\text{Rank}(d_1^{\bullet\bullet})) = 8 - 2 = 6, \quad \dim(\text{Rank}(d_2^{\bullet\bullet})) = 2 \quad (76)$$

we have

$$p_{de}^{\bullet\bullet}(T) \stackrel{(54)}{=} T^0(8 - 6) + T^1(8 - 6 - 2) + T^2(2 - 2) = 2 \quad (77)$$

and from the knowledge of q -dimensions in (75) we get:

$$P_{de}^{\bullet\bullet}(T, q) = q^2 \left(T^0(q^{-3} + q^{-1}) + (qT)^1 \cdot 0 + (qT)^2 \cdot 0 \right) = q^{-1} + q = D \quad (78)$$

i.e. coincides with the the answer (62) for the unknot, as necessary.

One can now deduce the same answer for the double-eight knot with three other colorings of vertices (black-white, white-black and white-white).

7.3.2 The $\bullet\circ$ case

Most interesting is the black-white case with

$$\mathcal{J}^{\bullet\circ} = p_1 p_4 + t(p_1 p_2 p'_1 + p_5) + t^2 p'_1 p'_4 \quad (79)$$

This is our first example, when differentials are not homogeneous:

$$\begin{aligned}
 d_1^{\bullet\circ} &= \vartheta_2 \vartheta'_1 \frac{\partial}{\partial \vartheta_4} + \vartheta_5 \frac{\partial^2}{\partial \vartheta_1 \partial \vartheta_4} = \theta_2 \theta'_1 \frac{\partial}{\partial \theta_4} + (\theta_2 \eta'_1 + \eta_2 \theta'_1) \frac{\partial}{\partial \eta_4} + \theta_5 \left(\frac{\partial^2}{\partial \eta_1 \partial \theta_4} + \frac{\partial^2}{\partial \theta_1 \partial \eta_4} \right) + \eta_5 \frac{\partial^2}{\partial \eta_1 \partial \eta_4}, \\
 d_2^{\bullet\circ} &= \vartheta'_4 \frac{\partial^2}{\partial \vartheta_1 \partial \vartheta_2} + \vartheta'_1 \vartheta'_4 \frac{\partial}{\partial \vartheta_5} = \theta'_4 \left(\frac{\partial^2}{\partial \eta_1 \partial \theta_2} + \frac{\partial^2}{\partial \theta_1 \partial \eta_2} \right) + \eta'_4 \frac{\partial^2}{\partial \eta_1 \partial \eta_2} + \theta'_1 \theta'_4 \frac{\partial}{\partial \theta_5} + (\theta'_1 \eta'_4 + \eta'_1 \theta'_4) \frac{\partial}{\partial \eta_5} \quad (80)
 \end{aligned}$$

Again, $d_2^{\bullet\circ} d_1^{\bullet\circ} = 0$, but kernels, images and even cohomologies are quite different from (75):

$$\begin{aligned} \text{Ker}(d_1^{\bullet\circ}) &= \emptyset, \implies \text{Rank}(d_1^{\bullet\circ}) = 4 \\ \text{Im}(d_1^{\bullet\circ}) &= \left\{ \theta_1 \theta_2 \theta'_1, (\theta_1 \eta_2 + \eta_1 \theta_2) \theta'_1 + \theta_1 \theta_2 \eta'_1, \theta_5 - \eta_1 \theta_2 \theta'_1, \eta_5 - \eta_1 (\theta_2 \eta'_1 + \eta_2 \theta'_1) \right\}, \\ \text{Coker}(d_2^{\bullet\circ}) &= \text{Im}(d_1^{\bullet\circ}) \oplus \left\{ (\theta_1 \eta_2 - \eta_1 \theta_2) \theta'_1, (\theta_1 \eta_2 - \eta_1 \theta_2) \eta'_1 \right\}, \\ \text{Coim}(d_2^{\bullet\circ}) &= \emptyset, \implies \text{Rank}(d_2^{\bullet\circ}) = 4 \end{aligned} \quad (81)$$

(note, that basis elements are not obligatory homogeneous). Thus

$$\begin{aligned} p_{de}^{\bullet\circ}(T) &= T^{-1} \left(T^0(2^2 - 4) + T^1(2^3 + 2 - 4 - 4) + T^2(4 - 4) \right) = 2, \\ P_{de}^{\bullet\circ}(T, q) &= \frac{q}{q^2 T} \left(T^0 \cdot 0 + (qT)(q + q^{-1}) + (qT)^2 \cdot 0 \right) = D \end{aligned} \quad (82)$$

in accordance with (78).

The white-black case is fully symmetric to black-white and does not bring anything new.

7.3.3 The $\circ\circ$ case

The white-white case, however, is not identical to black-black. Unlike it happened with the Hopf link in sec.7.2, the white-white cohomologies are now different from black-black (in a trivial way, of course, since the example is trivial) – only superpolynomial remains the same. Extended Jones polynomial

$$\mathcal{J}^{\bullet\bullet} = p_5 + t(p_1 p_4 + p'_1 p'_4) + t^2 p_1 p_2 p'_1 \quad (83)$$

implies that

$$\begin{aligned} d_1^{\circ\circ} &= (\vartheta_1 \vartheta_4 + \vartheta'_1 \vartheta'_4) \frac{\partial}{\partial \vartheta_5} = (\theta_1 \theta_4 + \theta'_1 \theta'_4) \frac{\partial}{\partial \theta_5} + (\theta_1 \eta_1 + \eta_1 \theta_1 + \theta'_1 \eta'_1 + \eta'_1 \theta'_1) \frac{\partial}{\partial \eta_5}, \\ d_2^{\circ\circ} &= \vartheta_1 \vartheta_2 \frac{\partial^2}{\partial \vartheta'_4} - \vartheta'_1 \vartheta_2 \frac{\partial}{\partial \vartheta_4} = \theta_1 \theta_2 \frac{\partial}{\partial \theta'_4} + (\theta_1 \eta_2 + \eta_1 \theta_2) \frac{\partial}{\partial \eta'_4} - \theta'_1 \theta_2 \frac{\partial}{\partial \theta_4} - (\theta'_1 \eta_2 + \eta'_1 \theta_2) \frac{\partial}{\partial \eta_4} \end{aligned} \quad (84)$$

Thus

$$\begin{aligned} \text{Ker}(d_1^{\circ\circ}) &= \emptyset, \implies \text{Rank}(d_1^{\circ\circ}) = 2 \\ \text{Im}(d_1^{\circ\circ}) &= \text{Ker}(d_2^{\circ\circ}) \implies \text{Rank}(d_2^{\circ\circ}) = 6 \\ \text{Coim}(d_2^{\circ\circ}) &= \left\{ -\theta_1 \eta_2 \eta'_1 + 2\eta_1 \theta_2 \eta'_1 - \eta_1 \eta_2 \theta'_1, \eta_1 \eta_2 \eta'_1 \right\}, \end{aligned} \quad (85)$$

so that

$$\begin{aligned} p_{de}^{\circ\circ}(T) &= T^{-2} \left(T^0(2 - 2) + T^1(2^2 + 2^2 - 2 - 6) + T^2(2^3 - 6) \right) = 2, \\ P_{de}^{\circ\circ}(T, q) &= \frac{1}{(q^2 T)^2} \left(T^0 \cdot 0 + (qT) \cdot 0 + (qT)^2 (q + q^3) \right) = D = P_{de}^{\bullet\bullet}(T, q) = P^{\text{unknot}}(T, q) \end{aligned} \quad (86)$$

As promised, this time $H_2^{\circ\circ} \cong V$, while in the "mirror case" non-vanishing was $H_0^{\bullet\bullet} \cong V$.

7.4 Example: Trefoil ($n = 3$)

Differentials are immediately written if we use \mathcal{Q} and formulas (30).

7.4.1 Three black vertices: the starting vertex is $r_c = [000] = p_3 p'_3$

Since, according to (30), the extended Jones polynomial in this case is

$$\mathcal{J}^{\bullet\bullet\bullet} = p_3 p'_3 + t(p_6 + p'_6 + p''_6) + t^2(p_4 p_2 + p'_4 p'_2 + p''_4 p''_2) + t^3 p_2 p'_2 p''_2,$$

we get for the differentials:

$$\begin{aligned}
d_1 &= (\vartheta_6 + \vartheta'_6 + \vartheta''_6) \frac{\partial^2}{\partial \vartheta_3 \partial \vartheta'_3}, \\
d_2 &= (\vartheta'_4 \vartheta'_2 - \vartheta''_4 \vartheta''_2) \frac{\partial}{\partial \vartheta_6} + (\vartheta''_4 \vartheta''_2 - \vartheta_4 \vartheta_2) \frac{\partial}{\partial \vartheta'_6} + (\vartheta_4 \vartheta_2 - \vartheta'_4 \vartheta'_2) \frac{\partial}{\partial \vartheta''_6}, \\
d_3 &= \vartheta_2 \vartheta'_2 \frac{\partial}{\partial \vartheta''_4} + \vartheta''_2 \vartheta_2 \frac{\partial}{\partial \vartheta'_4} + \vartheta'_2 \vartheta'_2 \frac{\partial}{\partial \vartheta_4}
\end{aligned} \tag{87}$$

Convolution with Q -tensors is implied, but suppressed (i.e. actually $\vartheta_a \vartheta_b \frac{\partial}{\partial \vartheta_c} = Q_{ij}^k \vartheta_a^i \vartheta_b^j \frac{\partial}{\partial \vartheta_c^k}$), the properties of Q allow cyclic permutations of θ 's. and this is all what is needed to prove that

$$d_3 d_2 = d_2 d_1 = 0 \tag{88}$$

We do not consider cohomologies in this case, because this will be done in more generality in s.7.5 below, and – by a more primitive method – in Appendix B at the end of the paper.

7.4.2 The case $\Gamma_c = \circ \bullet \bullet : r_c = [100] = p_6$

Since

$$\mathcal{J}^{\circ \bullet \bullet} = p_6 + t(p'_4 p'_2 + p''_4 p''_2 + p_3 p'_3) + t^2(p'_6 + p''_6 + p_2 p'_2 p''_2) + t^3 p_4 p_2,$$

we have:

$$\begin{aligned}
d_1 &= (\vartheta'_4 \vartheta'_2 + \vartheta''_4 \vartheta''_2 + \vartheta_3 \vartheta'_3) \frac{\partial}{\partial \vartheta_6}, \\
d_2 &= (\vartheta'_6 + \vartheta''_6) \frac{\partial^2}{\partial \vartheta_3 \partial \vartheta'_3} - \vartheta'_6 \frac{\partial^2}{\partial \vartheta''_4 \partial \vartheta'_2} - \vartheta''_6 \frac{\partial^2}{\partial \vartheta'_4 \partial \vartheta'_2} + \vartheta_2 \vartheta'_2 \frac{\partial}{\partial \vartheta'_4} + \vartheta_2 \vartheta'_2 \frac{\partial}{\partial \vartheta''_4}, \\
d_3 &= \vartheta_4 \vartheta_2 \left(\frac{\partial}{\partial \vartheta'_6} - \frac{\partial}{\partial \vartheta''_6} \right) + \vartheta_4 \frac{\partial^2}{\partial \vartheta'_2 \partial \vartheta''_2}
\end{aligned} \tag{89}$$

As in above example (7.3.2), the differentials are not homogeneous, thus their kernel and images have somewhat sophisticated bases.

7.5 Generic n , all vertices black

As in all previous examples of 2-strand braids, we label the cycles by index $a = (n_a, i)$, i.e. by its length and additional index i , enumerating different cycles of the same length.

The 2-strand braid with n vertices is a graph Γ with $2n$ edges, which connect pairwise the two subsequent vertices. Considering various resolutions of crossings at the vertices, it is clear that there are two cycles of length n and n cycles of each even length: $2, 4, 6, \dots, 2n$, i.e. there are two time-variables p_n, p'_n and also $p_{2k}^{(i)}$ with $k = 1, \dots, n$ and $i = 1, \dots, n$ (even if n is even, the cycles, γ_n, γ'_n are different from $\gamma_n^{(i)}$ and also different are the time-variables). If vertices are enumerated in a natural way, one can think of the cycle $p_{2k}^{(i)}$ as consisting of all edges of Γ between the vertices i and $i + k$.

Then it is clear that the extended Jones polynomial in this case is

$$\begin{aligned}
\mathcal{J}^{(n)} &= p_n p'_n + t \sum_{i=1}^n p_{2n}^{(i)} + t^2 \sum_{i < j} p_{2(j-i)}^{(i)} p_{2n-2(j-i)}^{(j)} + \dots + t^k \sum_{i_1 < i_2 < \dots < i_k} p_{2(i_2-i_1)}^{(i_1)} p_{2(i_3-i_2)}^{(i_2)} \dots p_{2n-2(i_k-i_1)}^{(i_k)} + \\
&\quad + \dots + t^n \prod_{i=1}^n p_2^{(i)} \tag{90}
\end{aligned}$$

From (90) the ordinary unreduced Jones polynomial is

$$J^{(n)} = q^n \left(D^2 - nqD + \frac{n(n-1)}{2} q^2 D^2 - \dots \right) = q^n \left(D^2 - 1 + (1 - qD)^n \right) = q^n \left(q^{-2} + 1 + q^2 + (-q^2)^n \right) \tag{91}$$

and consists of four items – this is a familiar formula, which we already derived in many ways in this text.

Moat important, from (90) we straightforwardly read the differentials.

7.5.1 Differential d_1

$$d_1 = \left(\sum_{i=1}^n \vartheta_{2n}^{(i)} \right) \frac{\partial^2}{\partial \vartheta_n \partial \vartheta'_n} \quad (92)$$

As usual for 2-strand knots, the d_1 is a product of two independent factors, what matters is the second one, which does not actually depend on n . Thus, as in all previous examples, the rank of d_1 is two, its kernel and its image are

$$\begin{aligned} \text{Ker}(d_1) &= \{ \theta_n \theta'_n, (\theta_n \eta'_n - \eta_n \theta'_n) \}, \quad \dim_q(H_0) = \dim_q \text{Ker}(d_1) = q^{-2} + 1, \\ \text{Im}(d_1) &= \left\{ \sum_{i=1}^n \theta_{2n}^{(i)}, \sum_{i=1}^n \eta_{(2n)}^{(i)} \right\}, \quad \dim_q \text{Im}(d_1) = q^{-1} + q = q^{-1} \left(\dim_q(C_0) - \dim_q \text{Ker}(d_1) \right) \end{aligned} \quad (93)$$

In the last equality the factor q^{-1} appears because our differentials decrease q -grading by one.

7.5.2 Differential d_2

$$d_2 = \sum_{i=1}^n \left(- \sum_{j>i} \vartheta_{2j-2i}^{(i)} \vartheta_{2n+2i-2j}^{(j)} + \sum_{j<i} \vartheta_{2i-2j}^{(j)} \vartheta_{2n+2j-2i}^{(i)} \right) \frac{\partial}{\partial \vartheta_{2n}^{(i)}} \quad (94)$$

The terms with $j > i$ in this sum are associated with the edges $[0 \dots 0 \overset{i}{1} 0 \dots 0 \overset{j}{\star} 0 \dots 0]$ of the hypercube, connecting its vertices $[0 \dots 0 \overset{i}{1} 0 \dots 0 \overset{j}{0} 0 \dots 0]$ (i.e. $p_{2n}^{(i)}$) and $[0 \dots 0 \overset{i}{1} 0 \dots 0 \overset{j}{1} 0 \dots 0]$ (i.e. $p_{2j-2i}^{(i)} p_{2n+2i-2j}^{(j)}$). Therefore, according to the rule (48) they enter with the sign factor $\epsilon_E = -1$. The terms with $j < i$ are associated with edges $[0 \dots 0 \overset{j}{\star} 0 \dots 0 \overset{i}{1} 0 \dots 0]$ between $[0 \dots 0 \overset{j}{0} 0 \dots 0 \overset{i}{1} 0 \dots 0]$ and $[0 \dots 0 \overset{j}{1} 0 \dots 0 \overset{i}{1} 0 \dots 0]$, and the sign factor is $+1$.

Clearly, $d_2 d_1 = 0$. The kernel of d_2 is non-vanishing only because the sum over of the bracket over i is zero, thus iff all derivatives $\frac{\partial}{\partial \vartheta_{2n}^{(i)}}$ are the same, the action of d_2 gives zero. This means that

$$\text{Ker}(d_2) = \left\{ \sum_{i=1}^n \theta_{2n}^{(i)}, \sum_{i=1}^n \eta_{2n}^{(i)} \right\}, \quad \dim_q \text{Ker}(d_2) = q^{-1} + q = D \quad (95)$$

Comparing with the second line of (93), we conclude that $\text{Ker}(d_2) = \text{Im}(d_1)$, and

$$H_1 = \text{Ker}(d_2) / \text{Im}(d_1) = \emptyset \quad (96)$$

As to the image of d_2 ,

$$\dim_q \text{Im}(d_2) = q^{-1} \left(\dim_q(C_1) - \dim_q \text{Ker}(d_2) \right) = (n-1) q^{-1} D \quad (97)$$

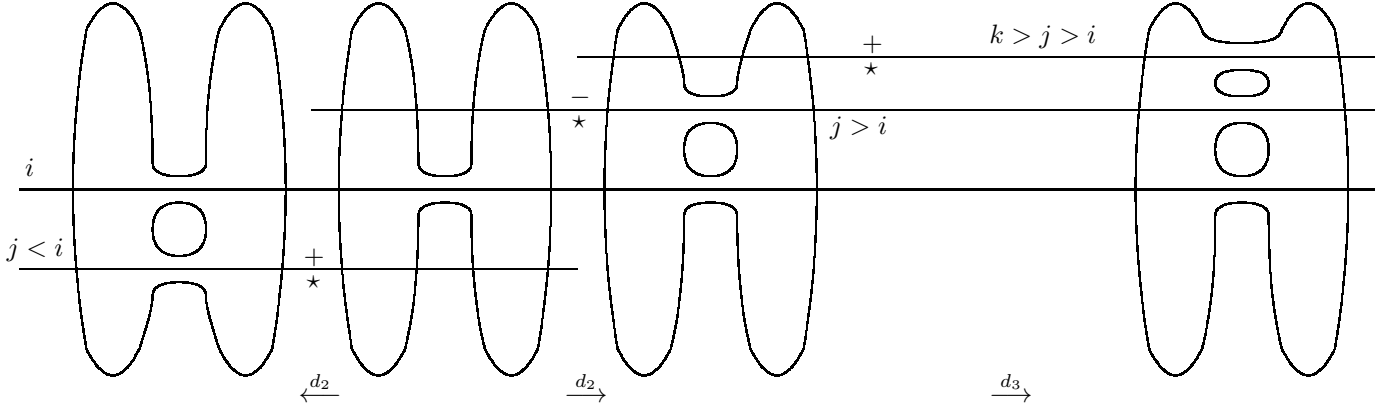
In fact, if we agree to define the cycle lengths modulo $2n$, i.e. define $\vartheta_{2k}^{(i)}$ with negative lengths $-2k$ as $\vartheta_{2n-2k}^{(i)}$, then d_2 acquires a simpler form:

$$d_2 = \sum_{i,j} \vartheta_{2j-2i}^{(i)} \vartheta_{2i-2j}^{(j)} \frac{\partial}{\partial \vartheta_{2n}^{(j)}} = - \sum_{i,j} \vartheta_{2j-2i}^{(i)} \vartheta_{2i-2j}^{(j)} \frac{\partial}{\partial \vartheta_{2n}^{(i)}} \quad (98)$$

(the terms with $i = j$ are automatically excluded by anticommutativity of ϑ -variables).

7.5.3 Differential d_3

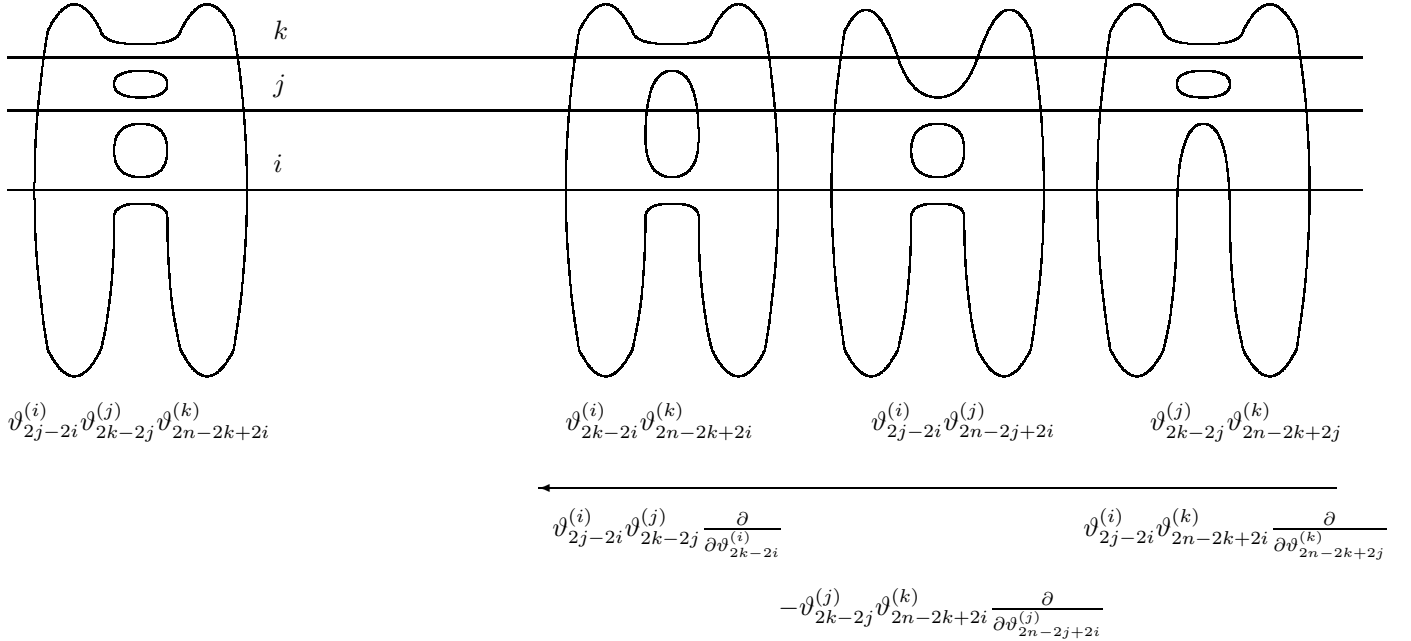
Pictorially the action of d_2 and d_3 can be represented as follows:



In this picture d_2 acts on the cycle $\gamma_{2n}^{(i)}$ of length $2n$, which is obtained by gluing γ_n and γ'_n at the vertex i . The action of d_2 makes a flip at the vertex j , this splits $\gamma_{2n}^{(i)}$ into two cycles: one of the length $2(j-i)$ between vertices i and j and a complementary one of the length $2n-2(j-i)$. The sign factor (48) depends on the sign $j-i$, this is also shown in the picture (of course, the sign depends on the conventions about the ordering of ϑ variables in (46)).

Operator d_3 acts in just the same way, making a new flip at vertex k . We show just one case of six possible orderings in the picture: $k > j > i$. In this case splitted is the cycle $\gamma_{2n-2j+2i}^{(j-i)}$ (complementary to the one between i and j), and produced are the two new cycles of the length $2(k-j)$ and $2n-2(k-i)$. The sign factor (48) in this case is positive.

The next picture shows three different elements from the space C_2 , which are mapped by d_3 into a given element of C_3 with $i < j < k \leq n$:



It is easy to see that – if the cycle lengths are calculated modulo $2n$ – that all the three cases are described by the single formula

$$d_3 = - \sum_{i,j,k} \vartheta_{2j-2i}^{(i)} \vartheta_{2k-2j}^{(j)} \frac{\partial}{\partial \vartheta_{2k-2i}^{(i)}} \stackrel{(46)}{=} - \sum_{i,j,k} \left(\theta_{2j-2i}^{(i)} \theta_{2k-2j}^{(j)} \frac{\partial}{\partial \vartheta_{2k-2i}^{(i)}} + \left(\theta_{2j-2i}^{(i)} \eta_{2k-2j}^{(j)} + \eta_{2j-2i}^{(i)} \theta_{2k-2j}^{(j)} \right) \frac{\partial}{\partial \eta_{2k-2i}^{(i)}} \right) \quad (99)$$

Already at the level of d_3 the cohomology calculus is getting rather tedious.

We begin it from the simplest case of $n = 3$. For this we explicitly describe the action of d_3 on the 12 elements of the basis in $C_2 = V_4 \otimes V_2 \oplus V'_4 \otimes V'_2 \oplus V''_4 \otimes V''_2$ (we remind that all vector spaces are isomorphic to

two-dimensional V , the labels are, however, important to define the action of d_3):

$$d_3^{(n=3)} : \begin{array}{c|c|c|c} \theta_2^{(i)} \theta_4^{(i+1)} & (i=1,2,3) \longrightarrow & \theta_2^{(1)} \theta_2^{(2)} \theta_2^{(3)} & 3 \longrightarrow 1 \quad 2q^{-2} \\ \hline \eta_2^{(i)} \eta_4^{(i+1)} & (i=1,2,3) \longrightarrow & \eta_2^{(i)} \left(\eta_2^{(i+1)} \theta_2^{(i+2)} + \theta_2^{(i+1)} \eta_2^{(i+2)} \right) & 3 \longrightarrow 3 \quad 0 \\ \hline \eta_2^{(i)} \theta_4^{(i+1)} & (i=1,2,3) \longrightarrow & \eta_2^{(i)} \theta_2^{(i+1)} \theta_2^{(i+2)} & 3 \longrightarrow 3 \quad 0 \\ \hline \theta_2^{(i)} \eta_4^{(i+1)} & (i=1,2,3) \longrightarrow & \theta_2^{(i)} \left(\eta_2^{(i+1)} \theta_2^{(i+2)} + \theta_2^{(i+1)} \eta_2^{(i+2)} \right) & 3 \nearrow \quad 3 \end{array} \quad (100)$$

The last column shows the contribution to $\dim_q \text{Ker}(d_3)$. Thus we see that

$$\begin{aligned} & \text{for } n=3 \quad \dim_q \text{Ker}(d_3) = 2q^{-2} + 3, \\ \implies \dim_q(H_2^{(3)}) &= \dim_q \text{Ker}(d_3) - \dim_q \text{Im}(d_2) \stackrel{(97)}{=} (2q^{-2} + 3) - 2q^{-1}D = 1 \end{aligned} \quad (101)$$

what implies that the contribution of the order T^2 to $P^{\bullet\bullet\bullet}(T, q)$ is $q^3(qT)^2 \cdot 1 = q^5 T^2$.

Next, $n=4$:

$$d_3^{(n=4)} : \begin{array}{c|c|c|c} \theta_2^{(i)} \theta_6^{(i+1)} & (i=1, \dots, 4) \longrightarrow & \theta_2^{(i)} \left(\theta_2^{(i+1)} \theta_4^{(i+2)} + \theta_4^{(i+1)} \theta_2^{(i+3)} \right) & 4 \longrightarrow 3 \quad q^{-3} \\ \hline \theta_4^{(i)} \theta_4^{(i+2)} & (i=1,2) \longrightarrow & \theta_2^{(i)} \theta_2^{(i+1)} \theta_4^{(i+2)} - \theta_4^{(i)} \theta_2^{(i+2)} \theta_2^{(i+3)} & 2 \nearrow \quad 2q^{-3} \\ \hline \eta_2^{(i)} \theta_6^{(i+1)} & (i=1, \dots, 4) \longrightarrow & \eta_2^{(i)} \left(\theta_2^{(i+1)} \theta_4^{(i+2)} + \theta_4^{(i+1)} \theta_2^{(i+3)} \right) & 4 \longrightarrow 4 \\ \hline \theta_2^{(i)} \eta_6^{(i+1)} & (i=1, \dots, 4) \longrightarrow & \theta_2^{(i)} \left(\eta_2^{(i+1)} \theta_4^{(i+2)} + \theta_2^{(i+1)} \eta_4^{(i+2)} + \right. & 4 \nearrow \\ & & \left. + \eta_4^{(i+1)} \theta_2^{(i+3)} + \theta_4^{(i+1)} \eta_2^{(i+3)} \right) & \searrow \quad 4 \\ \hline \eta_4^{(i)} \theta_4^{(i+2)} & (i=1, \dots, 4) \longrightarrow & \left(\eta_2^{(i)} \theta_2^{(i+1)} + \theta_2^{(i)} \eta_2^{(i+1)} \right) \theta_4^{(i+2)} - & 4 \longrightarrow 4 \\ & & - \eta_4^{(i)} \theta_2^{(i+2)} \theta_2^{(i+3)} & \\ \hline \eta_2^{(i)} \eta_6^{(i+1)} & (i=1, \dots, 4) \longrightarrow & \eta_2^{(i)} \left(\eta_2^{(i+1)} \theta_4^{(i+2)} + \eta_4^{(i+1)} \theta_2^{(i+3)} \right. & 4 \longrightarrow 4 \quad 0 \\ & & \left. + \theta_2^{(i+1)} \eta_4^{(i+2)} + \theta_4^{(i+1)} \eta_2^{(i+3)} \right) & \\ \hline \eta_4^{(i)} \eta_4^{(i+2)} & (i=1,2) \longrightarrow & \left(\eta_2^{(i)} \theta_2^{(i+1)} + \theta_2^{(i)} \eta_2^{(i+1)} \right) \eta_4^{(i+2)} & 2 \longrightarrow 2 \quad 0 \\ & & - \eta_4^{(i)} \left(\eta_2^{(i+2)} \theta_2^{(i+3)} + \theta_2^{(i+2)} \eta_2^{(i+3)} \right) & \end{array} \quad (102)$$

This time the numbers in the last column can need a more detailed explanation. To see what happens, we introduce a brief notation for four different structures, arising in the image of d_3 :

$$\begin{aligned} \theta_4^{(i)} \theta_2^{(i+2)} \theta_2^{(i+3)} &= (422)_i = \alpha_i, \\ \theta_4^{(i)} \eta_2^{(i+2)} \theta_2^{(i+3)} &= (4\bar{2}2)_i = \beta_i, \\ \theta_4^{(i)} \theta_2^{(i+2)} \eta_2^{(i+3)} &= (42\bar{2})_i = \gamma_i, \\ \eta_4^{(i)} \theta_2^{(i+2)} \theta_2^{(i+3)} &= (\bar{4}22)_i = \delta_i \end{aligned} \quad (103)$$

(the image of the $\eta\eta$ sector is simple, and does not require additional comments). Then, for example,

$$\begin{aligned} \theta_2^{(1)} \theta_6^{(2)} &\longrightarrow -\alpha_3 - \alpha_2 \\ \theta_4^{(1)} \theta_4^{(3)} &\longrightarrow \alpha_3 - \alpha_1 \\ \theta_6^{(1)} \theta_2^{(4)} = -\theta_2^{(4)} \theta_2^{(1)} &\longrightarrow \alpha_6 + \alpha_5 = \alpha_2 + \alpha_1 \end{aligned} \quad (104)$$

The third line is the same as the first one, with $i=1$ shifted to $i+3$. The sum of the three lines is the d_3 -transform of the d_2 -image of $\theta_8^{(1)}$, thus it should vanish (in the last column) – and, obviously, does so. Transforms of the six individual terms in the sector $\theta\theta$ are all of the form $\alpha_j + \alpha_{j+1}$ or $\alpha_j - \alpha_{j+2}$, i.e. all are linear combinations of the three independent structures, say, $\alpha_1 + \alpha_2$, $\alpha_2 + \alpha_3$ and $\alpha_3 + \alpha_4$. This gives the final quantity $3q^{-2}$ in the last column of the table (102).

Similarly, in the $\theta\eta$ sector the structures, emerging in the image are: $u_j = \beta_{j+1} + \gamma_j$, $v_j = \beta_j + \gamma_{j+1} + \delta_j + \delta_{j+1}$ and $w_j = \beta_j + \gamma_j - \delta_{j+2}$. However, from these twelve quantities only eight are linear independent. For example,

$$\begin{aligned} d_3\left(d_2(\eta_8^{(1)})\right) &= d_3\left(\eta_2^{(1)} \theta_6^{(2)} + \eta_4^{(1)} \theta_4^{(3)} + \eta_6^{(1)} \theta_4^{(4)} + \theta_2^{(1)} \eta_6^{(2)} + \theta_4^{(1)} \eta_4^{(3)} + \theta_6^{(1)} \eta_4^{(4)}\right) = \\ &= (-\beta_3 - \gamma_2) + (\beta_3 + \gamma_3 - \delta_1) + (\beta_1 + \gamma_2 + \delta_1 + \delta_2) - (\beta_2 + \gamma_3 + \delta_2 + \delta_3) - (\beta_1 + \gamma_1 - \delta_3) + (\beta_2 + \gamma_1) = 0 \end{aligned} \quad (105)$$

This seems to provide four elements of $\text{Ker}(d_3)$, but in fact there is one linear relation between the four, so actually only three remain – this was the reason why 3 appears as the $\theta\eta$ -sector contribution to $\dim_q \text{Ker}(d_2)$. However, d_3 has one extra zero mode. Indeed, all v -variables are expressed through u and w :

$$v_j = w_{j+1} + w_j - u_j + \sum_{k=1}^4 (u_k - w_k) \quad (106)$$

Thus

$$n = 4 : \quad \dim_q \text{Ker}(d_3) = 3q^{-2} + 4 \stackrel{(97)}{=} \dim_q \text{Im}(d_2) \implies \dim_q(H_2^{(4)}) = 1 \quad (107)$$

and the T^2 contribution to $P^{(4)}(T, q)$ is $q^4(qT)^2 \cdot 1 = q^6 T^2$.

The answer remains structurally the same for all other $n \geq 3$:

$$\dim_q \text{Ker}(d_3) = (n-1)q^{-2} + n \implies \dim_q(H_2^{(n)}) = 1 \quad (108)$$

the contribution to $P^{(n)}(T, q)$ is $q^n(qT)^2 = q^{n+2} T^2$ and

$$\begin{aligned} \dim_q \text{Im}(d_3) &= q^{-1} \left(\dim_q(C_2) - \dim_q \text{Ker}(d_3) \right) = q^{-1} \left(\frac{n(n-1)}{2} D^2 - (n-1)q^{-2} - n \right) = \\ &= \frac{(n-1)(n-2)}{2} q^{-3} + n(n-2)q^{-1} + \frac{n(n-1)}{2} q \end{aligned} \quad (109)$$

7.5.4 Differential d_I with generic I

Once (98) and (99) are known, it is straightforward to write down the full BRST operator for the case of arbitrary 2-strand braid with all black vertices:

$$\mathcal{Q}^{(n)} = \sum_{I=1}^n d_I^{(n)} = \left(\sum_{i=1}^n \vartheta_{2n}^{(i)} \right) \frac{\partial^2}{\partial \vartheta_n \partial \vartheta_n'} - \sum_{i,j,k} \vartheta_{2j-2i}^{(i)} \vartheta_{2k-2j}^{(j)} \frac{\partial}{\partial \vartheta_{2k-2i}^{(i)}} \quad (110)$$

The first term is the d_1 from (92), and the second term depends on n also through the agreement $\vartheta_{-2l}^{(i)} = \vartheta_{2n-2l}^{(i)}$. Particular differential d_I arises automatically when this operator is restricted to the relevant space C_{I-1} . Conditions $d_{I+1}d_I = 0$ follow from the nilpotency of $\mathcal{Q}^{(n)}$, $(\mathcal{Q}^{(n)})^2 = 0$, which, in turn, depends on the properties (41) and (42) of the tensors Q_{ij}^k and Q_k^{ij} , implicitly present in (110) through the convention (46). Given these properties, we can write:

$$\begin{aligned} (\mathcal{Q}^{(n)})^2 &= \sum_{i,j,k} \vartheta_{2j-2i}^{(i)} \vartheta_{2k-2j}^{(j)} \frac{\partial}{\partial \vartheta_{2k-2i}^{(i)}} \cdot \sum_{i',j',k'} \vartheta_{2j'-2i'}^{(i')} \vartheta_{2k'-2j'}^{(j')} \frac{\partial}{\partial \vartheta_{2k'-2i'}^{(i')}} = \\ &= \sum_{\substack{i,j,k \\ i',j',k'}} \left(\delta_i^{i'} \delta_{j'-i'}^{k-i} \cdot \vartheta_{2j-2i}^{(i)} \vartheta_{2k-2j}^{(j)} \vartheta_{2k'-2j'}^{(j')} \frac{\partial}{\partial \vartheta_{2k'-2i'}^{(i')}} - \delta_i^{j'} \delta_{k'-j'}^{k-i} \cdot \vartheta_{2j-2i}^{(i)} \vartheta_{2k-2j}^{(j)} \vartheta_{2j'-2i'}^{(i')} \frac{\partial}{\partial \vartheta_{2k'-2i'}^{(i')}} \right) = \\ &= \sum_{i,j,k,k'} \vartheta_{2j-2i}^{(i)} \vartheta_{2k-2j}^{(j)} \vartheta_{2k'-2k}^{(k)} \frac{\partial}{\partial \vartheta_{2k'-2i}^{(i)}} - \sum_{i,j,k,i'} \vartheta_{2j-2i}^{(i)} \vartheta_{2k-2j}^{(j)} \vartheta_{2i-2i'}^{(i')} \frac{\partial}{\partial \vartheta_{2k-2i'}^{(i')}} = 0 \end{aligned} \quad (111)$$

where at the last stage we make a change of summation variables in the second sum: $i' = i$, $k = k'$, $i = j$, $j = k$, which converts it into the first one.

For evaluation of cohomologies convenient is the second, more explicit, version of formula (99).

7.5.5 Differential d_n

d_n acts into 2^n -dimensional space with basis $\otimes_{i=1}^n \vartheta_2^{(i)}$ from $2^{m-1}n$ -dimensional one with the basis $\oplus_{i=1}^n \vartheta_4^{(i)} \otimes_{j \neq i, i+1} \vartheta_2^{(j)}$:

$$d_n = \sum_{i=1}^n \vartheta_2^{(i)} \vartheta_2^{(i+1)} \frac{\partial}{\partial \vartheta_4^{(i)}} = \sum_{i=1}^n \theta_2^{(i)} \theta_2^{(i+1)} \frac{\partial}{\partial \theta_4^{(i)}} + \sum_{i=1}^n (\theta_2^{(i)} \eta_2^{(i+1)} + \eta_2^{(i)} \theta_2^{(i+1)}) \frac{\partial}{\partial \eta_4^{(i)}} \quad (112)$$

Clearly, coimage of d_n contains $\prod_{i=1}^n \eta_2^{(i)}$ which has q -grading q^n . For n odd (knots) this is the only term in $\text{Coim}(d_n)$, however, for even n (links) there is one more: $\sum_{i=1}^n (-)^i \theta_2^{(i)} \prod_{j \neq i} \eta_2^{(j)}$. This means that the T^n -term in the superpolynomial is equal to

$$\begin{cases} 0 & \text{for } n = 1 \\ q^n(qT)^n q^n = q^{3n} T^n & \text{for odd } n > 1 \\ q^n(qT)^n(q^{n-2} + q^n) = q^{3n-2} T^n + q^{3n} T^n = q^{3n-1} D T^n & \text{for even } n \end{cases} \quad (113)$$

As to the T^{n-1} term, it is controlled by the kernel of d_n . In particular no η^{n-1} can appear in the image of d_{n-1} , while the action of d_n converts

$$d_n : \eta_4^{(i)} \prod_{k=2}^{n-1} \eta_2^{(i+k)} \longrightarrow \left(\theta_2^{(i)} \eta_2^{(i+1)} + \eta_2^{(i)} \theta_2^{(i+1)} \right) \prod_{k=2}^{n-1} \eta_2^{(i+k)} = \alpha_i - (-)^n \alpha_{i+1} \quad (114)$$

where $\alpha_i = \theta_2^{(i)} \prod_{k=2}^{n-1} \eta_2^{(i+k)}$. For even n variables there is one linear relation between the structures at the r.h.s.

$$\sum_{i=1}^n (\alpha_i - \alpha_{i+1}) = 0 \quad (115)$$

for odd n they are all linearly independent. This implies that the contribution of the order $q^n(qT)^{n-1}$ to the superpolynomial contains a q -grading factor q^{n-1} for even n , but not for odd n . In fact this means that for odd n the single contribution to H_{n-1} comes from the sector $\theta \eta^{n-2}$, and this factor is rather q^{n-3} .

7.5.6 Full answer

It turns out that in our example of 2-strand braids will all black vertices cohomologies $H_k^{(n)}$ with $1 < k < n$ are always 1-dimensional, and lie either in one of the two sectors: either $\eta^{k-1} \theta$ or η^k . Actually,

$$\dim_q(H_k^{(n)}) = \begin{cases} q^{k-2} & \text{for even } k \\ q^k & \text{for odd } k \end{cases} \quad (116)$$

Putting all pieces together, we finally obtain for the unreduced Jones superpolynomial in the fundamental representation for the torus knot/link of the type $[2, n]$:

$$\begin{aligned} P_{\square}^{(n)}(T, q) &\stackrel{[21, 3]}{=} q^n \left(q^{-1} D T^0 + 0 \cdot (qT)^1 + \sum_{\text{even } k=2}^{n-1} q^{k-2} (qT)^k + \sum_{\text{odd } k=3}^{n-1} q^k (qT)^k + \left(q^n q^{-1} D \right) (qT)^n \right) = \\ &= q^n \left(q^{-2} + 1 + q^2 T^2 + (1+T) \sum_{\text{odd } k=3}^{n-1} q^{2k} T^k + q^{2n} T^n \right) \end{aligned} \quad (117)$$

For $T = -1$ this turns back into the 4-term fundamental Jones (11). However, starting from $n = 4$ the unreduced Jones superpolynomial (117) acquires *more* terms than original Jones (more than four), there is a non-trivial contribution, proportional to $(T+1)$: this is a signal that new topological information occurs after T -deformation, not seen at all at $T \neq -1$. (Actually, the possibility to see this phenomenon at the level of 2-strand braids is the specifics of *non-reduced* knot polynomials. To observe it in the case of *reduced* polynomials, one should go beyond 2-strands.)

We can compare (117) with concrete answers, listed in [9]. They are given there for arbitrary N , we need $N = 2$ – this is the third column in the following table (we also omit lengthy expression for $n = 4$, while for $n = 6$ ref.[9] in any case gives it only for $N = 2, 3$; note also that torus links with $n = 4$ & 6 are labeled "v2" in [9] – "v1" differ by orientation of one of the two components of the link and are not torus):

n	$P^{(n)}(T, q N)$	$P^{(n)}(T, q)$	ref
2 :	$q^{N-1}[N] + (q^{2N}[N] - q^{N+1})[N]T^2$	$qD(1 + q^4 T^2)$	[4]
3 :	$q^{2(N-1)}([N] + [N-1]q^3 T^2(1 + q^{2N} T))$	$q + q^3 + q^5 T^2 + q^9 T^3$	[39]
4 :		$q^2 + q^4 + q^6 T^2 + q^{10} T^3 + q^{11} D T^4$	[9]
5 :	$q^{4(N-1)}([N] + [N-1]q^3 T^2(1 + q^{2N} T)(1 + q^4 T^2))$	$q^3 + q^5 + q^7 T^2 + q^{11} T^3 + q^{11} T^4 + q^{15} T^5$	[39]
6 :		$q^4 + q^6 + q^8 T^2 + q^{12} T^3 + q^{12} T^4 + q^{16} T^5 + q^{17} D T^6$	[9]
...			

7.6 A technical comment

Note that in all above examples Grassmannian nature of variables did not play too much *technical* role: it affects only signs, while nilpotency is not really important – it is automatically taken into account by the choice of the vector spaces to act from and to. This means that in cohomology calculations one can actually work directly with the cut-and-join operator (17), what is considerably easier in more complicated examples.

Also, in Appendix B at the end of this paper we give a more traditional description of the simplest examples, representing the differentials as explicit rectangular matrices – this, however, is a difficult approach if one needs to consider *series* of knots/links, say, all the 2-strand braids at once (what is needed, for example, to compare with the *evolution* method of [8]).

8 Conclusion

In this introductory review of Khovanov’s approach to the superpolynomials we discussed the basic example of the unreduced Jones superpolynomial and described in some detail the simplest case of the 2-strand knots. However, we did not yet manage to find an acceptably clear explanation (desirably free of the homological algebra techniques) of the topological invariance, which should separate the locality principle and local restrictions, imposed on differential operators by invariance under the Reidemeister moves.

The 2-strand example is sufficient to demonstrate the usual peculiarity of the homological approach: explicit construction of all the relevant spaces and operators is very tedious – but at the same time transparent and absolutely algorithmic. This makes it easy to get *answers* with the help of computer simulations (though computer abilities are exponentially decreasing with the growth of the crossing numbers), but pretty difficult to study any theoretical issues, like searching for hidden *structures*, getting one-parametric *families* of formulas etc – it is difficult to keep any parameters unfixed.

Construction includes the substitution of original **link diagram** (a 4-valent oriented graph with colored vertices of two types) by a **hypercube** of its resolutions, which is then promoted to a hypercube **quiver** and projected into a **complex** with differentials, made from the quiver maps. The superpolynomial is the **Poincare polynomial** if the complex. To construct the differentials in the complex we used the Koszul counterpart of the **cut-and-join operator** and **extended Jones polynomial**: looking at the latter one can pick up the appropriate items of the former. The main freedom in the construction is the quiver structure: the choice of the vector spaces at the hypercube vertices (associated with connected components of the graph resolutions) and the choice of the linear maps between them. We did not describe explicitly the restrictions, imposed on these spaces and maps by locality principle (none?) by appearance of the complex in the projection (commutativity and associativity) and by the Reidemeister invariance.

In the following parts of this review series we are going to make this part of the story more explicit and address a number of obvious issues:

- (i) Generalization to
 - arbitrary groups $Sl(N)$
 - arbitrary representations
- (ii) Lifting to universal polynomials
 - $N \longrightarrow A = q^N$
 - $T \longrightarrow t = -qT = q^\beta$
- (iii) Generalization to
 - *reduced* superpolynomials, admitting character decomposition in MacDonald dimensions [8],
 - their further lifting to *extended* superpolynomials, where dimensions are substituted by full MacDonald polynomials, depending on infinitely many "time-variables" $\{p_k\}$ [8, 43]. (these are invariants of braids only, not of knots/links – but this is already the case for some versions of Khovanov-Rozansky construction for $N > 2$).
- (iv) Connection to index theorems and their reformulations in terms of SUSY quantum mechanics *a la* [48].
- (v) Relations with other approaches to superpolynomial calculus.

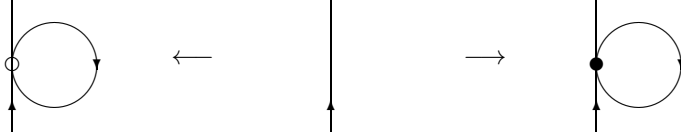
Appendix A: Sketch of invariance proof at the hypercube level

Each reformulation of Jones polynomial formula in sections 1-5 represents it in a different form, and, more important, in different terms. Therefore the proof or Reidemeister invariances at each level looks different and deserves separate discussion. Of course, existence of the most transparent proof in algebraic terms (sec.1) is sufficient, but, unfortunately, such formulation is not yet available for superpolynomials. On the other hand, relevant for superpolynomial deformation is reformulation in quiver terms (secs.3-4), which we did not yet

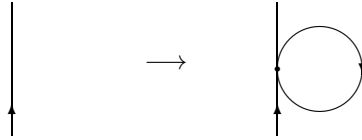
manage to present in adequate terms (for numerous homological-algebra presentations of the proof see [21]–[37]). Therefore we put accents away from invariance proofs in the main text. Still, in this appendix we describe a proof in terms of the intermediate stage: at the very important level of the hypercube of graph resolutions (sec.2). It helps to better understand, how the notion of *locality* – trivial at the algebraic level of sec.1 – gets non-trivial in all other formulations. Still at this level it is easy to present non-formally, what makes this intermediate example instructive and potentially useful.

***R1* move**

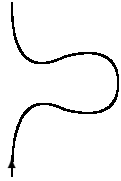
- Take a graph L with N vertices and a polynomial $J_L(q)$.
- Cut an edge and insert a "loop", with either a white or a black vertex:



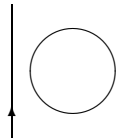
- For the graph Γ this means just insertion of a new uncolored vertex:



- Adding a vertex to a graph Γ means doubling the hypercube $\square(\Gamma)$.
The first copy contains a new resolved vertex of the type $=$:



The second copy – of the type $||$:



- The first insertion does not change $J_L(q)$, the second adds one new disconnected component to the graph, and multiplies $J_L(q)$ by γ .

- Another thing that changes is the initial vertex.

If we insert white vertex, then the first copy of the cube goes first – with the coefficient one, – while the second copy goes second – with the coefficient $-q$. And a new factor α is added.

If we insert the black vertex, then first – with the coefficient one – goes the second copy of the cube, while the first copy of the cube goes second – with the coefficient $-q$. And added is a factor β .

- Thus insertion of a loop with a white vertex substitutes the polynomial

$$J_L(q) \longrightarrow \alpha(1 - q\gamma)J_L(q) \quad (118)$$

while insertion of a loop with a black vertex –

$$J_L(q) \longrightarrow \beta(\gamma - q)J_L(q) \quad (119)$$

- Invariance under the Reidemeister move $R1$ requires that these two factors are unity:

$$\begin{aligned} \alpha(1 - q\gamma) &= 1, \\ \beta(\gamma - q) &= 1 \end{aligned} \quad (120)$$

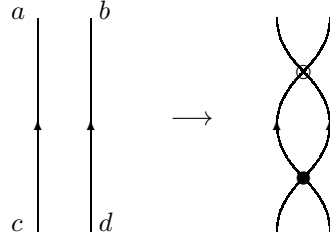
what expresses α and β through γ and q .

$R2$ move

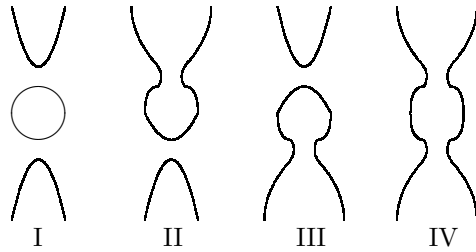
- Take a graph L with N vertices and a polynomial $J_L(q)$.
- Cut two edges. This provides a new graph $L_{ab|cd}$ with four external legs, two incoming (a, b) and two outgoing (c, d). Connecting c with a and d with b gives original graph $L_{ab|ab} = L$. Two other connections give new graphs: non-oriented $M = L_{aa|cc}$ and oriented $N = L_{ab|ba}$. The polynomials $J_M(q)$ and $J_N(q)$ in general are different from $J_L(q)$.

Now we can go in different directions, by making different kind of insertions between external legs. We begin with the Reidemeister move $R2$, and continue with the skein (Hecke algebra) relation in the next subsection.

- Insert a pair of a white and a black vertices:



- There are four different resolutions:



- Clearly, the first three resolutions will substitute $J_L(q)$ by $J_M(q)$, while the forth resolution will leave $J_L(q)$ intact. Also the first resolution adds one new disconnected component and thus a factor γ .

- Insertion of two white vertices means that the resolution I goes first (i.e. with coefficient one), the two resolutions II and III go second (i.e. with the coefficient $-q$), the resolution IV goes third (with the coefficient q^2). This means that

$$J_L(q) \longrightarrow \alpha^2 \left(\underbrace{\gamma J_M(q)}_I - q \cdot \underbrace{2J_M(q)}_{II+III} + q^2 \underbrace{J_L(q)}_{IV} \right) \quad (121)$$

Similarly, insertion of two black vertices would give

$$J_L(q) \longrightarrow \beta^2 \left(\underbrace{J_L(q)}_{IV} - q \cdot \underbrace{2J_M(q)}_{II+III} + q^2 \underbrace{\gamma J_M(q)}_I \right) \quad (122)$$

• Insertion one white and one black vertex at the first and second place respectively implies that the first – with the coefficient one – goes the resolution II , the second – with the coefficient $(-q)$ – go I and IV , and the last – with the coefficient q^2 – goes III , so that

$$J_L(q) \longrightarrow \alpha\beta \left(\underbrace{J_M(q)}_{II} - q \cdot \left(\underbrace{\gamma J_M(q)}_I + \underbrace{J_L(q)}_{IV} \right) + q^2 \underbrace{J_M(q)}_{III} \right) = -q\alpha\beta J_L(q) + \alpha\beta(1 - q\gamma + q^2) J_M(q) \quad (123)$$

Similarly, if the black vertex goes first and the white goes second, we have:

$$J_L(q) \longrightarrow \alpha\beta \left(\underbrace{J_M(q)}_{III} - q \cdot \left(\underbrace{\gamma J_M(q)}_I + \underbrace{J_L(q)}_{IV} \right) + q^2 \underbrace{J_M(q)}_{II} \right) = -q\alpha\beta J_L(q) + \alpha\beta(1 - q\gamma + q^2) J_M(q) \quad (124)$$

• Invariance under the $R2$ Reidemeister move implies that $J_L(q)$ remains intact when the pair of black and white vertices is inserted, this means that the coefficient in front of $J_L(q)$ should be one, and that in front of $J_M(q)$ vanishes:

$$\begin{aligned} -q\alpha\beta &= 1, \\ 1 - q\gamma + q^2 &= 0 \end{aligned} \quad (125)$$

Skein relation

• Going back, to the very beginning of the previous subsection, instead of a pair of vertices we could insert just one, and check the skein (or Hecke-algebra) relation

$$q^2 \circ -q^{-2} \bullet = q - q^{-1} \quad (126)$$

(this is the property of the \mathcal{R} -matrix in the fundamental representation).

• When white vertex is inserted, we get:

$$J_L(q) \longrightarrow \alpha(J_M(q) - qJ_L(q)) \quad (127)$$

When black vertex is inserted, we get instead

$$J_L(q) \longrightarrow \beta(J_L(q) - qJ_M(q)) \quad (128)$$

• Thus skein relation implies that

$$q^2\alpha(J_M(q) - qJ_L(q)) - q^{-2}\beta(J_L(q) - qJ_M(q)) = (q - q^{-1})J_L(q) \quad (129)$$

i.e.

$$\begin{aligned} \alpha q^2 + \beta q^{-1} &= 0, \\ -\alpha q^3 - \beta q^{-2} &= q - q^{-1} \end{aligned} \quad (130)$$

$R3$ move

- Take a graph L with N vertices and a polynomial $J_L(q)$.
- Cut three edges, obtain the graph L_{abcdef} with six ends, and insert a triple intersection in one of two ways:



In fact, the black vertex could also be white – what matters is that the two vertices on the horizontal line are of the same color.

- Each of the two pictures will have eight resolutions.
- When attached to L_{abcdef} these sixteen resolutions will give rise to just five new graphs,

$$A = L_{abddda}, \quad B = L_{abbaee}, \quad C = L_{abccba}, \quad D = L_{aacddc}, \quad E = L_{aaccee} \quad (131)$$

Equality between the two pictures (invariance under the $R3$ move, i.e. the Yang-Baxter relation) requires that coefficients coincide in front of A, B, C, D, E for both pictures. For B, C, D this happens automatically, and non-trivial are the conditions for A and E only, this gives two constraints:

$$\begin{aligned} 1 - q\gamma + q^2 &= 0, \\ q^2\gamma - q - q^3 &= 0 \end{aligned} \quad (132)$$

both satisfied for $\gamma = q + q^{-1}$.

• It is important here that there no graph L_{abcabc} appears in this process, this would provide one more constraint which would be impossible to satisfy. This does not allow to add a third type of resolution (crossing) into this whole construction.

Example of topological invariance: changing black to white in the 2-strand braid

If some m out of n black vertices in the 2-strand braid are changed for white, this provides a link, topologically equivalent to the one with $n - m$ black vertices only.

$n_{\circ} = 1, n_{\bullet} = n - 1$: Let us begin with the case of $m = 1$ and let us change the color of the vertex number I . Then this means that the starting vertex of the hypercube is now $\underline{1}000\dots 0$ instead of $0000\dots 0$. For convenience we underline this distinguished 1. The first-level vertices, instead of $0\dots 010\dots 0$ with a single unity at some place, are now substituted by $\underline{0}00\dots 0$ and $\underline{1}0\dots 010\dots 0$. Similarly, at level J , instead of the $\frac{n!}{J!(n-J)!}$ hypercube vertices with J unities, we now get $\frac{(n-1)!}{J!(n-1-J)!}$ vertices with unity at the first position and J unities somewhere else, as well as $\frac{(n-1)!}{(J-1)!(n-J)!}$ vertices with 0 at the first position and $J - 1$ unities somewhere else. Since in the case of 2-strand braids the number of connected components of resolved graph is equal to the number of unities in the label of hypercube vertex – with the single exception of $[00\dots 0]$ for which this number is two, we get:

$$\begin{aligned} J_{\square}^{n_{\circ}=1, n_{\bullet}=n-1}(q) &= -q^{-2} \cdot q^{n-1} \left\{ \underline{D} - q \left(\underline{(n-1)D^2} + \boxed{D^2} \right) + q^2 \left(\frac{1}{2} \underline{(n-1)(n-2)D^3} + (n-1)D \right) - \right. \\ &\quad \left. - q^3 \left(\frac{1}{6} \underline{(n-1)(n-2)(n-3)D^4} + \frac{1}{2} (n-1)(n-2)D^2 \right) + \dots \right\} = \\ &= -q^{n-3} \left\{ \underline{D(1-qD)^{n-1}} + \boxed{q(1-D^2)} - q(1-qD)^{n-1} \right\} = \\ &= -q^{n-3} \left\{ (D-q)(1-qD)^{n-1} - q(D^2-1) \right\} = q^{n-2} \left(q^2 + 1 + q^{-2} + (-q^2)^{n-2} \right) \stackrel{(11)}{=} J_{\square}^{(n-2)}(q) \quad (133) \end{aligned}$$

what is the right answer for Jones for the 2-strand braid with $n_{\bullet} - n_{\circ} = n - 2$ uncompensated crossings. The answer is composed from two series of items (one of them is underlined for convenience) and a "defect" (boxed), associated with "anomalous" hypercube vertex $[00\dots 0]$: it would smoothly get into the non-underlined series, if contributed $D^0 = 1$, but actually it contributes D^2 , and this defect should be explicitly taken into account.

$n_{\circ} = 2, n_{\bullet} = n - 2$: This time the starting vertex in the hypercube contains two units, let it be $\underline{11}00\dots 0$. At the first level we have: $\underline{1}000\dots 0$, $\underline{01}00\dots 0$ and $\underline{11}00\dots 010\dots 0$. At the second level appears the "anomalous" $\boxed{0000\dots 0}$ as well as $\underline{1}000\dots 010\dots 0$, $\underline{01}00\dots 010\dots 0$ and $\underline{11}00\dots 010\dots 010\dots 0$. At the third level we get $\underline{0000\dots 010\dots 0}$, $\underline{1000\dots 010\dots 010\dots 0}$, $\underline{0100\dots 010\dots 010\dots 0}$ and $\underline{1100\dots 010\dots 010\dots 010\dots 0}$ and so on. In result

$$\begin{aligned} J_{\square}^{n_{\circ}=2, n_{\bullet}=n-2}(q) &= q^{-4} \cdot q^{n-2} \left\{ \underline{D^2} - q \left(\underline{2D} + \underline{(n-2)D^3} \right) + q^2 \left(\boxed{D^2} + 2(n-2)D^2 + \frac{1}{2} \underline{(n-2)(n-3)D^4} \right) - \right. \\ &\quad \left. - q^3 \left((n-2)D + 2 \cdot \frac{1}{2} \underline{(n-2)(n-3)D^3} + \frac{1}{6} \underline{(n-2)(n-3)(n-4)D^5} \right) + \dots \right\} = \\ &= q^{n-6} \left\{ \underline{D^2(1-qD)^{n-2}} + \boxed{q^2(D^2-1)} - \underline{2qD(1-qD)^{n-2}} + q^2(1-qD)^{n-2} \right\} = \\ &= q^{n-6} \left\{ q^2(D^2-1) + (D-q)^2(1-qD)^{n-2} \right\} = q^{n-4} \left(q^2 + 1 + q^{-2} + (-q^2)^{n-4} \right) \stackrel{(11)}{=} J_{\square}^{(n-4)}(q) \quad (134) \end{aligned}$$

as it should be. Clearly, this time there are three series (two of them underlined once and twice respectively) and one defect term.

Now it is clear that for *arbitrary* n_\circ we obtain $n_\circ + 1$ series and the answer, implied by the hypercube pattern for the unreduced Jones polynomial, is:

$$\begin{aligned} J_{\square}^{n_\bullet = n - n_\circ}(q) &= (-)^{n_\circ} q^{n_\bullet - 2n_\circ} \left\{ (-q)^{n_\circ} (D^2 - 1) + (D - q)^{n_\circ} (1 - qD)^{n_\bullet} \right\} = \\ &= q^{n_\bullet - n_\circ} (q^2 + 1 + q^{-2} + (-q^2)^{n_\bullet - n_\circ}) \stackrel{(11)}{=} J_{\square}^{(n_\bullet - n_\circ)}(q) \end{aligned} \quad (135)$$

Appendix B: Examples of cohomology calculus at the level of matrices

We consider here the 2-strand knots with one, two and three intersections even more explicitly than it was done in ss.6 and 7.

Example: An eight

Let us take just one black vertex. Then hypercube is just a segment with two vertices $[0]$ and $[1]$, and

$$C_0 = V \otimes V, \quad C_1 = V \quad (136)$$

so that $J(q=1) = 2 \cdot 2 - 2 = 2$. The differential

$$d_1 = Q^* = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix} \quad (137)$$

It has rank two, therefore

$$P(T, q)|_{q=1} = (4 - 2)T^0 + (2 - 2)T^1 = 2 \quad (138)$$

The kernel of d_1 consists of two vectors $(01, -1, 0) = v_+ \otimes v_- - v_- \otimes v_+$ and $(0001) = v_- \otimes v_-$ with q -gradings 1 and q^{-2} respectively. The coimage of d_1 is empty. Therefore

$$P^\bullet(T, q) = q \left((1 + q^{-2})T^0 + 0 \cdot (qT) \right) = q + q^{-1} \quad (139)$$

If instead of black we would take a white vertex, the orientation of the hypercube (segment) quiver would be reversed,

$$\begin{aligned} C_0 &= V, \quad C_1 = V \otimes V, \\ J(q=1) &= -(2 - 2 \cdot 2) = 2 \end{aligned} \quad (140)$$

(the minus sign is because every white vertex enters with the coefficient $-q^{-2}$ instead of q for each black vertex). The differential

$$d_1 = Q = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \quad (141)$$

has the same rank 2 and

$$P(T, q)|_{q=1} = -((2 - 2)T^0 + (4 - 2)T^1) = -2T \quad (142)$$

Its kernel is empty, but coimage consists of two vectors (1000) and $(01, -10)$ with q -gradings q^2 and 1. Therefore

$$P^\circ(T, q) = T^{-1} q^{-2} (0 \cdot T^0 + (q^2 + 1)(qT)) = (q + q^{-1}) \quad (143)$$

Thus the two superpolynomials P^\bullet and P° coincide and are equal to the unreduced Jones superpolynomial for the unknot:

$$P^\bullet(T, q) = P^\circ(T, q) = P^{\text{unknot}}(T, q) = q + q^{-1} \quad (144)$$

Example: Hopf link

This is the case of two crossings with two vertices of the same color. Then

$$\begin{aligned} C_0 &= V \otimes V, & C_1 &= V \oplus V, & C_2 &= V \otimes V, \\ J(q=1) &= 2 \cdot 2 - (2+2) + 2 \cdot 2 = 4 \end{aligned} \quad (145)$$

Here

$$d_1 = \left(\begin{array}{c} Q^* \\ \hline Q^* \end{array} \right) = \left(\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{array} \right), \quad d_2 = \left(\begin{array}{c|c} Q & -Q \end{array} \right) = \left(\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{array} \right) \quad (146)$$

and the ranks and coranks of both operators are equal to 2, so that

$$P(T, q)|_{q=1} = (4-2)T^0 + (4-2-2)T^1 + (4-2)T^2 = 2(1+T^2) \quad (147)$$

Explicitly the zero-modes of these operators and their q -gradings are:

$$\begin{aligned} \text{Ker}(d_1) : & \quad (01, -10) \ \& \ (0001) \quad 1 \ \& \ q^{-2} \\ \text{Im}(d_1) : & \quad (10|10) \ \& \ (01|01) \quad q \ \& \ q^{-1} \\ \text{Ker}(d_2) : & \quad (10|10) \ \& \ (01|01) \quad q \ \& \ q^{-1} \\ \text{CoIm}(d_2) : & \quad (01, -10) \ \& \ (1000) \quad 1 \ \& \ q^2 \end{aligned} \quad (148)$$

Therefore

$$P^{\text{Hopf}}(T, q) = q^2 \left((1+q^{-2})T^0 + 0 \cdot (qT) + (1+q^2) \cdot (qT)^2 \right) = (1+q^2)(1+q^4T^2) \quad (149)$$

If two white vertices were taken instead of the two black ones, the only difference would be the change of overall factor from q^2 to $(Tq^2)^{-2}$, i.e. instead of (149) we obtain

$$T^{-2}q^{-6}(1+q^2)(1+q^4T^2) = P^{\text{Hopf}}(q^{-1}, T^{-1}) \quad (150)$$

If we now consider a pair of vertices of different colors, then

$$\begin{aligned} C_0 &= V, & C_1 &= (V \otimes V) \oplus (V \otimes V), & C_2 &= V, \\ J(q=1) &= (2 - 2 \times 2^2 + 2) = 4, \end{aligned} \quad (151)$$

Here

$$d_1 = \left(\begin{array}{c} Q \\ \hline Q \end{array} \right) = \left(\begin{array}{cc|cc} 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{array} \right), \quad (152)$$

and

$$d_2 = \left(\begin{array}{c|c} Q^* & -Q^* \end{array} \right) = \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & -1 & -1 & 0 \end{array} \right) \quad (153)$$

Both these matrices have rank 2, therefore

$$P(T, q)|_{q=1} = T^{-1} \left((2-2)T^0 + (8-2-2)T^1 + (2-2)T^2 \right) = 4 \quad (154)$$

Since $\text{Ker}(d_1) = \text{CoIm}(d_2) = \emptyset$, the only non-trivial is the cohomology H_1 . The image of d_1 consists of two vectors $(0110|0110)$ and $(0001|0001)$, while the kernel of d_2 – of six: $(abcd|aefg)$, provided $b+c=e+f$. This leaves in the cohomology H_1 just four vectors: $(1000|0000)$, $(0100|0100)$, $(0100|0010)$, $(0001|0000)$ with the q -gradings $q^2, 1, 1, q^{-2}$ respectively. Therefore

$$P(T, q) = \frac{q}{Tq^2} (q^2 + 2 + q^{-2})(qT) = (q + q^{-1})^2 = \left(P^{\text{unknot}}(T, q) \right)^2 \quad (155)$$

Example: Trefoil

We represent trefoil as a 2-strand braid with 3 black vertices. Then

$$\begin{aligned}
C_0 &= V \otimes V, \quad C_1 = V \oplus V \oplus V, \quad C_2 = (V \otimes V) \oplus (V \otimes V) \oplus (V \otimes V), \quad C_3 = V \otimes V \otimes V \\
J(q=1) &= 2 \cdot 2 - (2 + 2 + 2) + 3(2 \cdot 2) - 2 \cdot 2 \cdot 2 = 2, \\
P(q=1, T) &= \boxed{2} + (\underline{2} - \underline{2}) \cdot T + (\underline{5} - \underline{4}) \cdot T^2 + \boxed{1} \cdot T^3 = 2 + T^2 + T^3 \\
P(q, T) &= q + q^3 + q^5 T^2 + q^9 T^3
\end{aligned} \tag{156}$$

This time

$$d_1 = \begin{pmatrix} \frac{Q^*}{Q^*} \\ \frac{Q^*}{Q^*} \\ \frac{Q^*}{Q^*} \end{pmatrix}, \quad d_2 = \begin{pmatrix} 0 & Q & -Q \\ \hline Q & 0 & -Q \\ \hline Q & -Q & 0 \end{pmatrix}, \tag{157}$$

and

$$d_3 = \begin{pmatrix} & & & & V_{23}^{\otimes 2} & & & & V_{13}^{\otimes 2} & & & & V_{12}^{\otimes 2} \\ & & & & ++ & +- & -+ & -- & ++ & +- & -+ & -- & ++ & +- & -+ & -- \\ \hline +++ & 0 & 0 & & 0 & 0 & & & 0 & 0 & & & 0 & 0 & & \\ ++- & -1 & 0 & & 1 & 0 & & & & & & & & 0 & 0 & \\ +-+ & -1 & 0 & & & & & & 0 & 0 & & & -1 & 0 & & \\ +-- & 0 & -1 & & & & & & 1 & 0 & & & & -1 & 0 & \\ \hline -++ & & & & 0 & 0 & & 1 & 0 & & & & -1 & 0 & & \\ -+- & & & & -1 & 0 & & 0 & 1 & & & & & -1 & 0 & \\ --+ & & & & -1 & 0 & & & 1 & 0 & & 0 & -1 & & & \\ --- & & & & 0 & -1 & & & 0 & 1 & & & & 0 & -1 & \end{pmatrix} \tag{158}$$

so that $d_3 d_2 = d_2 d_1 = 0$.

For tensor product

$$\text{rank}(A \otimes B) = \text{rank}(A) \cdot \text{rank}(B) \tag{159}$$

Therefore the ranks and coranks of the product matrices are:

product	size	rank	corank
$d_1 \tilde{d}_1$	6×6	$\underline{2}$	4
$\tilde{d}_1 d_1$	4×4	2	$\boxed{2}$
$d_2 \tilde{d}_2$	12×12	$\underline{4}$	8
$\tilde{d}_2 d_2$	6×6	4	$\underline{2}$
$d_3 \tilde{d}_3$	8×8	7	$\boxed{1}$
$\tilde{d}_3 d_3$	12×12	7	$\underline{5}$

(160)

It remains to restore the q -dependence.

The zero-vectors of $\tilde{d}_1 d_1$ are $v_- \otimes v_-$ and $v_+ \otimes v_- - v_- \otimes v_+$, which have weights $(q^{-1})^2 = q^{-2}$ and $q^{-1} \cdot q = 1$ respectively.

The coimage of d_3 is $v_+ \otimes v_+ \otimes v_+$ and has the weight $q^3 \times q^3$ where the second q^3 is because it is in the third term in the complex.

Cohomology H_1 is empty. Indeed, $\text{Ker}(d_2)$ is spanned by two vectors: (v_+, v_+, v_+) and (v_-, v_-, v_-) (which would have weights $q \cdot q = q^2$ and $q^{-1} \cdot q = 1$ respectively) and they both belong to the image of d_1 .

Finally, the one-dimensional cohomology $H_2 = \text{Ker}(d_3)/\text{Im}(d_2)$ is made from the vector

$$v_+^1 \otimes v_-^{23} - v_-^1 \otimes v_+^{23} + 2v_+^2 \otimes v_-^{13} + v_+^3 \otimes v_-^{12} - v_-^3 \otimes v_+^{12} \tag{161}$$

which has weight $q^0 \cdot q^2$.

Thus the superpolynomial is

$$P(q, T) = q^3 \left((q^{-2} + 1) + q^2 T^2 + q^6 T^3 \right) = q + q^3 + q^5 T^2 + q^9 T^3 \tag{162}$$

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